Discrete Structures EPFL, Fall 2014

Solution to Problem Set 9

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Problem 1.

(a) **Base step.** Let $n = 1$. Since $\sum_{i=1}^{1} f_i^2 = f_i^2 = 1$ and $f_1 f_2 = 1$, then the assertion is true.

Induction step. Assume that $\sum_{i=1}^{n} f_i^2 = f_n \cdot f_{n+1}$ for some $n \geq 1$. Then, using the induction hypothesis and the fact that $f_{n+2} = f_{n+1} + f_n$, we obtain that

$$\sum_{i=1}^{n+1} f_i^2 = \sum_{i=1}^{n} f_i^2 + f_{n+1}^2 = f_n \cdot f_{n+1} + f_{n+1}^2 = f_{n+1}(f_n + f_{n+1}) = f_{n+1} \cdot f_{n+2}$$

(b) **Base step.** Let $n = 1$. Since $f_2 f_0 - f_1^2 = 0 - 1 = -1$ and $(-1)^1 = -1$, then the assertion is true.

Induction step. Assume that $f_{n+1} \cdot f_{n-1} - f_n^2 = (-1)^n$ for some $n \geq 1$. Using the recursive definition $f_{n+2} = f_{n+1} + f_n$, $f_{n+1} = f_n + f_{n-1}$ and applying the induction hypothesis, we obtain

$$\begin{aligned} f_{n+2} f_n - f_{n+1}^2 &= (f_{n+1} + f_n) f_n - f_{n+1}(f_n + f_{n-1}) \\ &= f_{n+1} f_n + f_n^2 - f_{n+1} f_n - f_{n+1} f_{n-1} \\ &= -(f_{n+1} f_{n-1} - f_n^2) \\ &= (-1) \cdot (-1)^n \\ &= (-1)^{n+1}. \end{aligned}$$

(c) **Base step.** Let $n = 1$. Observing that $\begin{bmatrix} f_2 & f_1 \\ f_1 & f_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and recalling the definition of A, we easily check the equality.

Induction step. Assume that $A^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}$. Then, using the recursive definition $f_{n+2} = f_{n+1} + f_n, f_{n+1} = f_n + f_{n-1}$ and applying the induction hypothesis, we obtain

$$A^{n+1} = A^n A = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} f_{n+1} + f_n & f_{n+1} \\ f_n + f_{n-1} & f_n \end{bmatrix} = \begin{bmatrix} f_{n+2} & f_{n+1} \\ f_{n+1} & f_n \end{bmatrix}.$$

Problem 2.

Let a_n be the answer to the problem and $n = 3^k$. We have the following recursion : $a_1 = 1$, $a_{3^k} = 3^k + a_{3^{k-1}}$, which follows immediately from the structure of the algorithm.

Claim: For all $k \geq 0$ we have that $a_{3^k} = \frac{3^{k+1} - 1}{2}$.

Proof by induction.

Base step: $a_{3^0} = 1$. [And indeed, if $n = 1$, we print the sentence once.]

Induction step: Suppose that for $k \geq 0$ it is true that $a_{3^k} = \dfrac{3^{k+1}-1}{2}$. Then, using the recursion, we obtain $a_{3^{k+1}} = 3^{k+1} + \dfrac{3^{k+1}-1}{2} = \dfrac{3^{k+2}-1}{2}$.

As a result, the phrase is printed $\dfrac{3^{k+1}-1}{2} = \dfrac{3n-1}{2}$ times.

Problem 3. Let v be a vertex of the regular convex n-gon. The number of diagonals departing from v is $n-3$, because there are $n-3$ vertices which are not adjacent to v. If we go over the n vertices and we sum the number of diagonals departing from each of them, we count every diagonal twice. Hence, the total number of diagonals is $\dfrac{n(n-3)}{2}$.

Problem 4.

(a) The first 3 letters are fixed, while the remaining 5 are free. Hence, this is equivalent to counting the strings of length 5 where each letter can take 26 different values. The total number of such strings is 26^5.

(b) We fix the first 2 and the last 2 letters, while the remaining 4 are free. Therefore, there are 26^4 possible strings.

(c) The number of strings which begin with `ab` is 26^6, the number of strings which end with `yz` is 26^6 and the number of strings which begin with `ab` and end with `yz` is 26^4. By inclusion-exclusion principle the number of strings which begin with `ab` or end with `yz` is $26^6 + 26^6 - 26^4$.

(d) First of all, let us count the ways in which one can choose the positions of these four `q`'s. These are the combinations of 4 elements from a class of 12 without replacement, which means that there are $\binom{8}{4}$ different ways of chooosing the positions of the `q`'s. The remaining four positions are occupied by any of the 25 letters which is different from a `q`. As a result, the total number of strings is $\binom{8}{4} \cdot 25^4$.

(e) (*Bonus.*) As concerns the first part of the question, counting the bit strings with exactly 4 1's is equivalent to counting the ways in which one can choose the positions of these four 1's. These are the combinations of 4 elements from a class of 12 without replacement. Therefore the solution is $\binom{12}{4}$.

As concerns the second part of the question, let A be the set of bit strings of length 12 which have exactly four 1's such that none of these 1's are adjacent to each other. Let B be the set of bit strings of length 9 with exactly four 1's.
Consider a string $a \in A$: it has four 1's which are not adjacent to each other. Hence, each of the first three 1's has to be followed by a 0. Consider the function $f : A \to B$ which maps the string $a \in A$ into the string $b \in B$ such that b is obtained removing the three 0's immediately after each of the first three 1's of a. It is easy to check that the function f is a bijection. Consequently, so $|A| = |B|$. By reasons similar to those of point (d), $|B| = \binom{9}{4}$ and, as a result, the solution is $\binom{9}{4}$.