Discrete Structures EPFL, Fall 2014

Solution to Problem Set 12

Date: 5.12.2014 Not graded

Problem 1.

(a) The set of divisors of 6 is $\{1, 2, 3, 6\}$. Thus, 6 has 4 divisors.

(b) The set of divisors of 36 is $\{1, 2, 3, 4, 6, 9, 12, 18, 36\}$. Hence, 36 has 9 divisors.

(c) In general, if d divides m, the prime factorization of d must contain the same prime factors each of them with at most the same power as in the factorization of m. More precisely, the prime factorization of d should be in the form of

$$d = \prod_{i=1}^{k} p_i^{\beta_i}, \qquad 0 \leq \beta_i \leq \alpha_i, \forall i = 1, \ldots, k.$$

Therefore, we can enumerate all the divisors of m by enumerating all the possible choices for $\beta_1, \ldots, \beta_k$. We have $1 + \alpha_1$ choices for β_1 (it can be picked from the set $\{0, 1, 2, \ldots, \alpha_1\}$), $1 + \alpha_2$ choices for β_2 and so on. Thus, the number of divisors of m is

$$\prod_{i=1}^{k} (1 + \alpha_i).$$

Problem 2.

(a) The first coin/bill deposited can be one of three currencies in hand: There are a_{n-1} ways to pay if the sequence begins with a \$1 coin; there are also a_{n-1} ways to pay if the sequence begins with a \$1 bill; and there are a_{n-2} ways to pay if the sequence begins with a \$2 bill. Thus we can write the recursion

$$a_n = 2a_{n-1} + a_{n-2}, \qquad \text{for } n \geq 2.$$

The initial conditions are $a_0 = 1$ (there is only one way to pay nothing) and $a_1 = 2$ (1 dollar can be paid either by coin or by bill).

(b) Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function associated to the sequence a_n and note that $x^k F(x) = \sum_{n=k}^{\infty} a_{n-k} x^n$. We thus have

$$\begin{aligned} F(x) - 2xF(x) - x^2 F(x) &= \sum_{n=0}^{\infty} a_n x^n - 2\sum_{n=0}^{\infty} a_{n-1} x^n - \sum_{n=0}^{\infty} a_{n-2} x^n \\ &= a_0 + a_1 x - 2a_0 x + \sum_{n=2}^{\infty} (a_n - 2a_{n-1} - a_{n-2}) x^n \\ &= 1 + 2x - 2x = 1. \end{aligned}$$

Consequently,

$$
\begin{aligned}
F(x) &= \frac{1}{1-2x-x^2} \\
&= \frac{1/2\sqrt{2}}{1+\sqrt{2}+x} - \frac{1/2\sqrt{2}}{1-\sqrt{2}+x} \\
&= \frac{1}{2\sqrt{2}+4} \times \frac{1}{1+\frac{x}{1+\sqrt{2}}} - \frac{1}{2\sqrt{2}-4} \times \frac{1}{1+\frac{x}{1-\sqrt{2}}} \\
&= \frac{1}{2\sqrt{2}+4} \sum_{n=0}^{\infty} \left(-\frac{1}{1+\sqrt{2}}\right)^n x^n - \frac{1}{2\sqrt{2}-4} \sum_{n=0}^{\infty} \left(-\frac{1}{1-\sqrt{2}}\right)^n x^n \\
&= \frac{1}{4}(2-\sqrt{2}) \sum_{n=0}^{\infty} (1-\sqrt{2})^n x^n + \frac{1}{4}(2+\sqrt{2}) \sum_{n=0}^{\infty} (1+\sqrt{2})^n x^n.
\end{aligned}
$$

Hence the sequence a_n is

$$a_n = \frac{1}{4}(2+\sqrt{2})(1+\sqrt{2})^n + \frac{1}{4}(2-\sqrt{2})(1-\sqrt{2})^n$$

Remark. Observe that the solution of the recursion is in the form $\alpha p^n + \beta q^n$ where p and q are the inverses of the roots of the denominator of the generating function, that is, the roots of the polynomial $r^2 - 2r - 1 = 0$ (this is obtained by replacing x with $1/r$ in the denominator of $F(x)$). This equation is called the characteristic function of the recursion and is obtained by first rewriting the recursion $a_{n+2} = 2a_{n+1} + a_n$ (shifting the indices so that the smallest index in the recurrence relationship is n) and then replacing a_{n+k} by r^k, $k = 0, 1, \ldots$. After finding the roots we then solve for α and β using the initial conditions.

(c) Suppose we choose to use n_1 \$1 coins, n_2 \$1 bills, and n_3 \$2 bills. Then b_n is the number of solutions of

$$n_1 + n_2 + 2n_3 = n, \qquad n_1, n_2, n_3 \in \mathbb{N}_{\geq 0}.$$

It is not difficult to see that b_n is the coefficient of x^n in

$$G(x) = (1 + x + x^2 + \ldots)(1 + x + x^2 + \ldots)(1 + x^2 + x^4 + \ldots) = \frac{1}{(1-x)^2} \frac{1}{1-x^2}.$$

In fact, we will get x^n in the result of the multiplication of the three polynomials whenever we multiply monomials x^{n_1}, x^{n_2} and x^{2n_3} such that $n_1 + n_2 + 2n_3 = n$ from the three polynomials respectively. Hence, the coefficient of x^n is equal to the number of different ways we can pick such monomials from the three polynomials.

By using partial fraction expansion, there are numbers α, β, γ, and δ such that

$$
\begin{aligned}
G(x) &= \frac{\alpha}{1-x} + \frac{\beta}{(1-x)^2} + \frac{\gamma}{(1-x)^3} + \frac{\delta}{1+x} \\
&= \frac{\alpha(1-x)^2(1+x) + \beta(1-x^2) + \gamma(1+x) + \delta(1-x)^3}{(1-x)^3(1+x)}.
\end{aligned}
$$

Setting the polynomial in the numerator equal to 1 we obtain the system of equations

$$
\begin{cases}
\alpha + \beta + \gamma + \delta = 1 \\
-\alpha + \gamma - 3\delta = 0 \\
-\alpha - \beta + 3\delta = 0 \\
\alpha - \delta = 0
\end{cases},
$$

whose solution (by elimination, for example) gives $\alpha = \delta = 1/8$, $\beta = 1/4$ and $\gamma = 1/2$.

We retrieve

$$\begin{aligned}
G(x) &= \frac{1/8}{1-x} + \frac{1/4}{(1-x)^2} + \frac{1/2}{(1-x)^3} + \frac{1/8}{1+x} \\
&= \frac{1}{8}\sum_{n=0}^{\infty} x^n + \frac{1}{4}\sum_{n=0}^{\infty}(n+1)x^n + \frac{1}{2}\sum_{n=0}^{\infty}\binom{n+2}{2}x^n + \frac{1}{8}\sum_{n=0}^{\infty}(-1)^n x^n \\
&= \sum_{n=0}^{\infty} x^n \left(\frac{1}{8}((-1)^n+1) + \frac{1}{4}(n+1) + \frac{1}{2}\binom{n+2}{2}\right),
\end{aligned}$$

from where we can read off each coefficient. So the answer is

$$b_n = \frac{1}{8}((-1)^n+1) + \frac{1}{4}(n+1) + \frac{1}{2}\binom{n+2}{2} = \left\lceil \frac{(n+3)(n+1)}{4} \right\rceil.$$

Problem 3. We know from the course that if $F(x)$ is the generating function associated with the sequence a_n and $b_n = \sum_{i=0}^{n} a_i$, then the generating function of the sequence b_n, which we denote by $G(x)$, is

$$G(x) = \frac{F(x)}{1-x}.$$

(a) $a_n = \Theta\left((\frac{1}{2})^n\right)$ means the closest root of the denominator (aka *pole*) of $F(x)$ to the origin is $x^* = 2$. In other words,

$$F(x) = \frac{P(x)}{(1-\frac{1}{2}x)\tilde{Q}(x)}$$

where all roots of $\tilde{Q}(x)$ have absolute value bigger than 2. Therefore,

$$G(x) = \frac{F(x)}{1-x} = \frac{P(x)}{(1-x)(1-\frac{1}{2}x)\tilde{Q}(x)}$$

has a pole $x^* = 1$ which is the closest pole to the origin. As a consequence we can conclude that

$$b_n = \Theta(1).$$

(b) Repeating the same argument, we know that

$$F(x) = \frac{P(x)}{(1-2x)\tilde{Q}(x)}$$

has the pole with the smallest magnitude $x^* = \frac{1}{2}$ (i.e. the roots of $\tilde{Q}(x)$ all have magnitude bigger than $\frac{1}{2}$). Consequently,

$$G(x) = \frac{P(x)}{(1-x)(1-2x)\tilde{Q}(x)}$$

still has its smallest-magnitude pole at $x^* = \frac{1}{2}$. This means

$$b_n = \Theta(2^n).$$

Problem 4. Let $G(x) = \sum_{n=0}^{\infty} b_n x^n$ be the generating function associated to b_n.

(a)

$$\begin{aligned}
G(x) &= \sum_{n=0}^{\infty}(2a_n - a_{n+1})x^n \\
&= 2\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_{n+1}x^n \\
&= 2F(x) - x^{-1}\sum_{n=0}^{\infty} a_{n+1}x^{n+1} \\
&= 2F(x) - x^{-1}\left(\sum_{n=0}^{\infty} a_n x^n - a_0\right) \\
&= 2F(x) - x^{-1}\left(F(x) - a_0\right) \\
&= \left(2 - \frac{1}{x}\right)F(x) + \frac{a_0}{x}.
\end{aligned}$$

(b)

$$\begin{aligned}
G(x) &= \sum_{n=0}^{\infty} n a_n x^n \\
&= x\sum_{n=0}^{\infty} n a_n x^{n-1} \\
&= x\frac{\partial}{\partial x}\left\{\sum_{n=0}^{\infty} a_n x^n\right\} \\
&= xF'(x)
\end{aligned}$$

(c)

$$\begin{aligned}
G(x) &= b_0 + \sum_{n=1}^{\infty} \frac{a_n}{n}x^n \\
&= b_0 + \sum_{n=0}^{\infty} \frac{a_{n+1}}{n+1}x^{n+1} \\
&= b_0 + \sum_{n=0}^{\infty} \int_0^x a_{n+1}t^n \mathrm{d}t \\
&= b_0 + \int_0^x \sum_{n=0}^{\infty} a_{n+1}t^n \mathrm{d}t \\
&= b_0 + \int_0^x t^{-1}\sum_{n=0}^{\infty} a_{n+1}t^{n+1}\mathrm{d}t \\
&= b_0 + \int_0^x t^{-1}\left(\sum_{n=0}^{\infty} a_n t^n - a_0\right)\mathrm{d}t \\
&= b_0 + \int_0^x t^{-1}\left(F(t) - a_0\right)\mathrm{d}t \\
&= a_0 + \int_0^x \frac{F(t) - a_0}{t}\mathrm{d}t
\end{aligned}$$

Problem 5.

(a) The probability that a fair coin lands Heads in a single trial is $\frac{1}{2}$. As the $2n$ trials are independent and in each of them the probability of obtaining Heads is $\frac{1}{2}$, the probability of having $2n$ Heads is $\frac{1}{2^{2n}}$.

Another way to see the result is the following. There are 2^{2n} possible outcomes which are all equiprobable. Only 1 consists of $2n$ Heads. Hence, the required probability is $\frac{1}{2^{2n}}$.

(b) There are 2^{2n} possible outcomes which are all equiprobable. In $\binom{2n}{2}$ of them, we obtain 2 Tails. Hence, the required probability is $\frac{\binom{2n}{2}}{2^{2n}}$.

(c) Generalizing the argument above, we have that the probability of observing k Tails is given by

$$p_k = \frac{\binom{2n}{k}}{2^{2n}}.$$

We should bet on k^* such that p_k attains its maximum at k^*. Let us consider the ratio $\frac{p_{k+1}}{p_k}$. After some simplifications, we obtain

$$\frac{p_k + 1}{p_k} = \frac{2n - k}{k + 1} \geq 1 \iff k \leq n - \frac{1}{2}.$$

Recalling that k and n are integers, we deduce that $p_{k+1} \geq p_k$ for $k < n$ and that $p_{k+1} \leq p_k$ for $k \geq n$. Hence, p_k attains its maximum when $k^* = n$ and you should bet on n Tails.

(d) The probability of getting Heads is $1 - \frac{1}{3} = \frac{2}{3}$. As the $2n$ trials are independent and in each of them the probability of obtaining Heads is $\frac{2}{3}$, the probability of having $2n$ Heads is $\left(\frac{2}{3}\right)^{2n}$. In addition, in $\binom{2n}{2}$ cases, we obtain 2 Tails and each of these cases occurs with probability $\left(\frac{2}{3}\right)^{2n-2} \times \left(\frac{1}{3}\right)^2$. Hence, the probability of getting 2 Tails out of $2n$ trials is $\frac{2^{2n-2}\binom{2n}{2}}{3^{2n}}$.

Problem 6.

(a) The probability of getting a sum equal to 2 is $p_1 q_1$. This probability is also equal to $\frac{1}{11}$, since the sum of the outcomes is uniform in $\{2, 3, \cdots, 12\}$. Therefore,

$$p_1 q_1 = \frac{1}{11}. \tag{1}$$

(b) The probability of getting a sum equal to 12 is $p_6 q_6$. This probability is also equal to $\frac{1}{11}$, since the sum of the outcomes is uniform in $\{2, 3, \cdots, 12\}$. Therefore,

$$p_6 q_6 = \frac{1}{11}.$$

(c)

$$\frac{a + b}{2} \geq \sqrt{ab} \iff a + b \geq 2\sqrt{ab} \iff (a + b)^2 \geq 4ab,$$

where the last $\iff$ is allowed because a and b are non-negative and, therefore, their sum is non-negative. In addition,

$$(a + b)^2 - 4ab = (a - b)^2 \geq 0,$$

which is enough to prove the desired inequality.

(d) Let s be the probability that the sum of the outcomes of the two dice is 7. Then,

$$s = p_1q_6 + p_2q_5 + p_3q_4 + p_4q_3 + p_5q_2 + p_6q_1 \geq p_1q_6 + p_6q_1 = \frac{1}{11}\left(\frac{p_1}{p_6} + \frac{p_6}{p_1}\right),$$

where the last equality comes from points (a) and (b). Using part (c), we also obtain that

$$\frac{p_1}{p_6} + \frac{p_6}{p_1} \geq 2\sqrt{\frac{p_1}{p_6} \cdot \frac{p_6}{p_1}} = 2.$$

Consequently, $s \geq \frac{2}{11}$. In addition, s must be equal to $\frac{1}{11}$, because the sum of the outcomes is uniform in $\{2, 3, \cdots, 12\}$, from which we obtain a contradiction.

(e) The generating function of the sum of independent random variables is the product of the individual generating functions, i.e., $s(x) = p(x)q(x)$. In addition, using the fact that the sum of outcomes is uniform in $\{2, 3, \cdots, 12\}$, we obtain that

$$s(x) = \frac{1}{11}\sum_{i=2}^{12} x^i = \frac{1}{11} \cdot x^2 \cdot \frac{x^{11} - 1}{x - 1}.$$

The polynomial $\frac{x^{11}-1}{x-1}$ has no real roots. Indeed, the fact that $x = 1$ is not a root can be easily checked by euclidean division. In addition, $x = 1$ is the only real root of $x^{11} - 1$. On the other hand, $p(x)$ is equal to x times a polynomial of odd degree and the same reasoning applies to $q(x)$. Hence, $s(x) = p(x)q(x)$ has two roots in 0 plus two other real roots, which is a contradiction.