11. Differentials and the chain rule

Let $w = f(x, y, z)$ be a function of three variables. Introduce a new object, called the total differential.

$$\mathrm{d}f = f_x \,\mathrm{d}x + f_y \,\mathrm{d}y + f_z \,\mathrm{d}z.$$

Formally behaves similarly to how Δf behaves,

$$\Delta f \approx f_x \,\Delta x + f_y \,\Delta y + f_z \,\Delta z.$$

However it is a new object (it is not the same as a small change in f as the book would claim), with its own rules of manipulation. For us, the main use of the total differential will be to understand the chain rule.

Suppose that x, y and z are functions of one variable t. Then $w = f(x, y, z)$ becomes a function of t. Divide the equation above to get the derivative of f,

$$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt}.$$

This is an instance of the chain rule.

Example 11.1. *Let $f(x, y, z) = xyz + z^2$. Suppose that $x = t^2$, $y = 3/t$ and $z = \sin t$.*

Then

$$f_x = yz \qquad f_y = xz \qquad \text{and} \qquad f_z = 2z,$$

so that

$$\frac{dw}{dt} = 2yzt - \frac{3xz}{t^2} + (xy + 2z)\cos t = 3\sin t + (3t + 2\sin t)\cos t.$$

On the other hand, if we substitute for x, y and z, we get

$$w = 3t\sin t + \sin^2 t,$$

and we can calculate directly,

$$\frac{dw}{dt} = 3\sin t + 3t\cos t + 2\sin t\cos t.$$

There are two ways to see that the chain rule is correct.

$$\mathrm{d}x = x'(t)\,\mathrm{d}t \qquad \mathrm{d}y = y'(t)\,\mathrm{d}t \qquad \text{and} \qquad \mathrm{d}z = z'(t)\,\mathrm{d}t.$$

Substituting we get

$$\begin{aligned}\mathrm{d}w &= f_x \,\mathrm{d}x + f_y \,\mathrm{d}y + f_z \,\mathrm{d}z \\ &= f_x x'(t)\,\mathrm{d}t + f_y y'(t)\,\mathrm{d}t + f_z z'(t)\,\mathrm{d}t,\end{aligned}$$

and dividing by $\mathrm{d}t$ gives us the chain rule.

More rigorously, start with the approximation formula,

$$\Delta w \approx f_x \,\Delta x + f_y \,\Delta y + f_z \,\Delta z,$$

divide both sides by Δt and take the limit as $\Delta t \to 0$.

One can use the chain rule to justify some of the well-known formulae for differentiation.

Let $f(u,v) = uv$. Suppose that $u = u(t)$ and $v = v(t)$ are both functions of t. Then

$$\frac{d(uv)}{dt} = f_u \frac{du}{dt} + f_v \frac{dv}{dt} = vu' + uv',$$

which is the product rule. Similarly if $f = u/v$, then

$$\frac{d(u/v)}{dt} = f_u \frac{du}{dt} + f_v \frac{dv}{dt} = \frac{1}{v}u' - \frac{u}{v^2}v' = \frac{u'v - v'u}{v^2},$$

which is the quotient rule.

Now suppose that $w = f(x,y)$ and $x = x(u,v)$ and $y = y(u,v)$. Then

$$\begin{aligned} \mathrm{d}w &= f_x \,\mathrm{d}x + f_y \,\mathrm{d}y \\ &= f_x(x_u \,\mathrm{d}u + x_v \,\mathrm{d}v) + f_y(y_u \,\mathrm{d}u + y_v \,\mathrm{d}v) \\ &= (f_x x_u + f_y y_u)\,\mathrm{d}u + (f_x x_v + f_y y_v)\,\mathrm{d}v. \\ &= f_u \,\mathrm{d}u + f_v \,\mathrm{d}v. \end{aligned}$$

If we write this out in long form, we have

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \qquad \text{and} \qquad \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

Example 11.2. *Suppose that $w = f(x,y)$ and we change from Cartesian to polar coordinates,*

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta. \end{aligned}$$

We have

$$\begin{aligned} \frac{\partial x}{\partial r} &= \cos\theta & \qquad \frac{\partial x}{\partial \theta} &= -r\sin\theta \\ \frac{\partial y}{\partial r} &= \sin\theta & \qquad \frac{\partial x}{\partial \theta} &= r\cos\theta. \end{aligned}$$

So

$$\begin{aligned} f_r &= \cos\theta f_x + \sin\theta f_y \\ f_\theta &= -r\sin\theta f_x + r\cos\theta f_y. \end{aligned}$$