12. The Gradient and directional derivatives

We have
$$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt}.$$
We can rewrite this as
$$\nabla f \cdot \vec{v}(t),$$
where
$$\nabla f = f_x \hat{\imath} + f_y \hat{\jmath} + f_z \hat{k} \qquad \text{and} \qquad \vec{v} = \frac{d\vec{r}}{dt} = x'(t)\hat{\imath} + y'(t)\hat{\jmath} + z'(t)\hat{k}/$$

∇f is called the gradient of f. For a given point, we get a vector (so that ∇f is a vector valued function). Perhaps one of the most important properties of the gradient is:

Theorem 12.1. *∇f is orthogonal to the level surface $w = c$.*

Example 12.2. *Let $f(x, y, z) = ax + by + cz$.*

The level surface $w = d$ is the plane
$$ax + by + cz = d.$$
The gradient is
$$\nabla f = \langle a, b, c \rangle,$$
which is indeed a normal vector to the plane $ax + by + cz = d$.

Example 12.3. *Let $f(x, y) = x^2 + y^2$.*

The level curve $w = c$ is a circle,
$$x^2 + y^2 = c,$$
centred at the origin of radius $\sqrt{c}$. The gradient is
$$\nabla f = \langle 2x, 2y \rangle,$$
which is a radial vector, orthogonal to the circle.

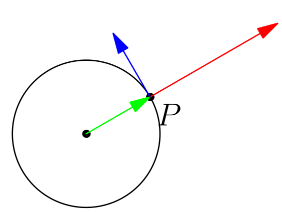

Figure 1. 3 vectors: green position, red gradient, blue velocity

Example 12.4. *Let $f(x, y) = y^2 - x^2$.*

The level curve is a hyperbola,

$$y^2 - x^2 = c,$$

with asymptotes $y = x$ and $y = -x$. The gradient is

$$\nabla f = \langle -2x, 2y \rangle.$$

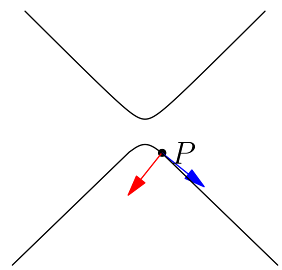

FIGURE 2. Red gradient, blue tangent vector

Proof of (**??**). Pick a curve $\vec{r}(t)$ contained in the level surface $w = c$. The velocity vector $\vec{v} = \vec{r}'(t)$ is contained in the tangent plane. By the chain rule,

$$0 = \frac{dw}{dt} = \nabla f \cdot \vec{v} = 0,$$

so that ∇f is perpendicular to every vector parallel to the tangent plane. $\square$

We can use this to calculate the tangent plane. For example, consider

$$2x^2 - y^2 - z^2 = 6.$$

Let's calculate the tangent plane to this surface at the point $(x_0, y_0, z_0) = (2, 1, 1)$. We have

$$\nabla f = \langle 4x, -2y, -2z \rangle.$$

At $(x_0, y_0, z_0) = (2, 1, 1)$, the gradient is $\langle 8, -2, -2 \rangle$, so that $\vec{n} = \langle 4, -1, -1 \rangle$ is a normal vector to the tangent plane. It follows that the equation of the tangent plane is

$$0 = \langle x - 2, y - 1, z - 1 \rangle \cdot \langle 4, -1, -1 \rangle \qquad \text{so that} \qquad 4x - y - z = 6,$$

is the equation of the tangent plane.

In this example, there are other ways to figure out an equation for the tangent plane. We could write z as a function of x and y,

$$z = \sqrt{2x^2 - y^2 - 6},$$

and find an equation for the tangent plane in the standard way. Beware that this is not always possible.

Suppose that we are at a point (x_0, y_0) in the plane and we move in a direction $\hat{u} = \langle a, b \rangle$. We can define the directional derivative in the direction $\hat{u}$. Consider the line

$$\vec{r}(s) = \langle x_0, y_0 \rangle + s\langle a, b \rangle.$$

The velocity vector is $\hat{u}$, which has unit length, so that the speed is one. In other words, $\vec{r}(s)$ is parametrised by arclength.

$$\left.\frac{dw}{ds}\right|_{\hat{u}} = \lim_{s \to 0} \frac{f(x_0 + sa, y_0 + sb) - f(x_0, y_0)}{\Delta s}.$$

If $\hat{u} = \hat{\imath}$, then the directional derivative is f_x and if $\hat{u} = \hat{\jmath}$ then the directional derivative is f_y. In general, if we slice the graph $w = f(x, y)$ by vertical planes, the directional derivative is the slope of the resulting curve.

$$\left.\frac{dw}{ds}\right|_{\hat{u}} = \nabla f \cdot \frac{d\vec{r}}{ds} = \nabla f \cdot \hat{u}.$$

Question 12.5. *Fix a vector $\vec{v} = \langle c, d \rangle$ in the plane. Which unit vector $\hat{u}$*

(1) maximises $\vec{v} \cdot \hat{u}$?
(2) minimises $\vec{v} \cdot \hat{u}$?
(3) When is $\vec{v} \cdot \hat{u} = 0$?

We know

$$\vec{v} \cdot \hat{u} = |\vec{v}||\hat{u}| \cos\theta = |\vec{v}| \cos\theta.$$

$|\vec{v}|$ is fixed as $\vec{v}$ is fixed. So we want to

(1) maximise $\cos\theta$,
(2) minimise $\cos\theta$
(3) and we want to know when $\cos\theta = 0$.

This happens when

(1) $\theta = 0$, in which case $\cos\theta = 1$,
(2) $\theta = \pi$, in which case $\cos\theta = -1$,
(3) and $\theta = \pi/2$, in which case $\cos\theta = 0$.

Geometrically the three cases correspond to:

(1) $\hat{u}$ points in the same direction as $\vec{v}$,
(2) $\hat{u}$ points in the opposite direction, and
(3) $\hat{u}$ is orthogonal to $\vec{v}$.

Now consider $v = \nabla f$. The directional derivative is

$$\left.\frac{dw}{ds}\right|_{\hat{u}} = \nabla f \cdot \hat{u}.$$

This is maximised when $\hat{u}$ points in the direction of ∇f. In other words, ∇f points in the direction of maximal increase, $-\nabla f$ points in the direction of maximal decrease and it is orthogonal to the level curves. The magnitude $|\nabla f|$ of the gradient is the directional derivative in the direction of ∇f, it is the largest possible rate of change.

In terms of someone climbing a mountain: ∇f points in the direction you need to go straight up the mountain, with magnitude the slope. $-\nabla f$ points straight down and ∇f is orthogonal to the level curve, which is the direction which takes you around the mountain.