13. Lagrange multipliers

If we want to maximise a function over a region, we also need to maximise the function over the boundary of the region, which is often given to us as a level surface. We are asked to solve something like

$$\text{maximise} \quad f(x, y, z) \qquad \text{subject to} \qquad g(x, y, z) = c.$$

Typical problem:

Example 13.1. *Find the closest point to the origin lying on the hyperbola $xy = 4$.*

If we put this in the form above we want to

$$\text{minimise} \quad x^2 + y^2 \qquad \text{subject to} \qquad xy = 4.$$

One obvious way to solve any such problem is to eliminate a variable.

$$y = \frac{4}{x}.$$

So we want to minimise

$$x^2 + \frac{16}{x^2}.$$

It turns out that in even quite simple examples, this can get very messy.

Now we know that one of the closest points is $(2, 2)$, and at this point the tangent directions to the curve $xy = 4$ and to the curve $x^2 + y^2 = 8$ (the level curve of the function we want to maximise) are parallel.

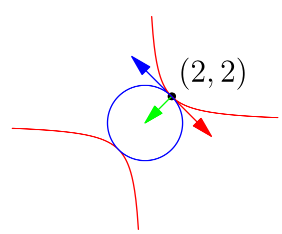

Figure 1. Closest point: tangent vectors are parallel

If the tangent vectors are not parallel, we can do better, we can move along the curve $xy = 4$ to a closer point. Here is what happens at $(4, 1)$:

In other words, at the closest point, the normal directions to the level curves are parallel, so that ∇f and ∇g are parallel, that is, there is a scalar λ, called a multiplier, such that

$$\nabla f = \lambda \nabla g.$$

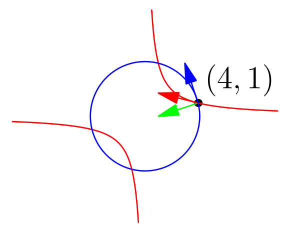

FIGURE 2. Tangent vectors are not parallel: we can do better

This suggests a strategy to solve optimisation problems. Introduce a new variable λ. We want x, y and λ such that

$$\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ g &= c. \end{aligned}$$

In our case this means

$$\begin{aligned} 2x &= \lambda y \\ 2y &= \lambda x \\ xy &= 4. \end{aligned}$$

Rearranging, this gives

$$\begin{aligned} 2x - \lambda y &= 0 \\ -\lambda x + 2y &= 0. \end{aligned}$$

A homogeneous system of linear equations. Either $(x, y) = (0, 0)$, which is not possible since $xy = 4$. Or the determinant is zero,

$$0 = \begin{vmatrix} 2 & -\lambda \\ -\lambda & 4 \end{vmatrix} = 4 - \lambda^2.$$

So $\lambda = \pm 2$. If $\lambda = -2$ then x and y have different sign, impossible. So $\lambda = 2$ is the only solution, in which case $x = y$. But then $x^2 = 4$ and $x = y = 2$ (or -2).

Why does this work? Well, if we are at a maximum of f subject to the constraint $g = c$, then if we move in any direction $\hat{u}$ in the surface $g = c$, we must have

$$0 = \left.\frac{df}{ds}\right|_{\hat{u}} = \nabla f \cdot \hat{u}.$$

It follows that ∇f is orthogonal to the level surface $g = c$. As ∇g also has this property, ∇f and ∇g are parallel, so that there is a multiplier.

Warning: The second derivative test won't work if one uses Lagrange multipliers. You just have to look at the critical points and compare them to see which is the maximum and which is the minimum.

Example 13.2. *What are the dimensions of a box with largest volume if the total surface area is* 64*?*

Let x, y and z be the sides of the box. The surface area of the box is

$$2(xy + xz + yz).$$

So we want to

$$\text{maximise} \quad xyz \qquad \text{subject to} \qquad xy + xz + yz = 32.$$

Let's first try to solve this problem without Lagrange multipliers. Use the constraint to express z as a function of x and y,

$$z = \frac{32 - xy}{x + y}.$$

Plug this back into the volume,

$$\text{maximise} \quad \frac{xy(32 - xy)}{x + y}.$$

The next step would be to find the partials with respect to x and y. A mess!

Instead, let's use Lagrange multipliers. Introduce λ. We want to solve $\nabla f = \lambda \nabla g$, so that

$$\begin{aligned} yz &= \lambda(y + z) \\ xz &= \lambda(x + z) \\ yz &= \lambda(x + y) \\ xy + xz + yz &= 32. \end{aligned}$$

Multiply the first equation by x, the second by y and the third by z.

$$\begin{aligned} xyz &= \lambda x(y + z) \\ xyz &= \lambda y(x + z) \\ xyz &= \lambda z(x + y) \\ xy + xz + yz &= 32. \end{aligned}$$

So

$$\lambda x(y + z) = \lambda y(x + z).$$

Suppose that $\lambda = 0$. Then $yz = 0$ so that one of y or z is zero. Similarly one of x and y is zero and one of x and z is zero. So two out of three of x, y and z would be zero, impossible.

If $\lambda \neq 0$, we get

$$xy + xz = yx + yz.$$

So

$$xz = yz.$$

Again, $z \neq 0$, so $x = y$. By symmetry, $x = y = z$. Hence

$$3x^2 = 32.$$

Hence

$$x = \frac{\sqrt{32}}{\sqrt{3}}.$$

Let's check that a cube is a maximum (and not a minimum or a saddle point). We cannot use the 2nd derivative test. We just have to think about it. What happens if we increase x? Since the surface area is constant, y and z must decrease to zero and the volume goes to zero. Similarly as we increase either y or z. Since we approach zero as we go to the boundary, the cube has to correspond to a maximum.