# 21. Potential Functions

Suppose that $\vec{F} = M\hat{\imath} + N\hat{\jmath} = \nabla f$ is a gradient vector field. Then

$$M_y = f_{xy} = f_{yx} = N_x.$$

So, if $\vec{F}$ is a gradient vector field then $M_y = N_x$.

**Theorem 21.1.** *Let $\vec{F} = M\hat{\imath} + N\hat{\jmath}$ be a vector field which is defined and differentiable on the whole of $\mathbb{R}^2$.*

*Then $\vec{F}$ is a gradient vector field if and only if $M_y = N_x$.*

**Example 21.2.** *Let $\vec{F} = -y\hat{\imath} + x\hat{\jmath}$. Then $M = -y$ and $N = x$. So*

$$M_y = -1 \qquad \text{and} \qquad N_x = 1,$$

*which are not equal. So $\vec{F}$ is not a gradient vector field.*

**Question 21.3.** *For which values of $a$ is $\vec{F} = (4x^2+axy)\hat{\imath}+(3y^2+4x^2)\hat{\jmath}$ a gradient field?*

We have

$$M = 4x^2 + axy \qquad \text{and} \qquad N = 3y^2 + 4x^2.$$

So

$$M_y = ax \qquad \text{and} \qquad N_x = 8x.$$

It follows that $M_y = N_x$ if and only if $a = 8$.

Given that (21.1) is true, it follows that if $M_y = N_x$ then $\vec{F} = \nabla f$, for some scalar function $f(x, y)$. We give two methods to calculate $f$, when

$$\vec{F} = (4x^2 + 8xy)\hat{\imath} + (3y^2 + 4x^2)\hat{\jmath}.$$

**Method 1:** We could use the fundamental theorem of calculus for line integrals. Suppose we want to determine the value of $f(x, y)$ at a point $(x_1, y_1)$. Pick a curve $C$ starting at $(0, 0)$ and ending at $(x_1, y_1)$. We have

$$f(x_1, y_1) - f(0, 0) = \int_C \vec{F} \cdot \mathrm{d}\vec{r}.$$

Note $f(0, 0)$ is an integration constant. If $f$ is a potential function then so is $f + c$.

Note that we get to choose $C$. A sensible choice in this example is to decompose $C$ as the straight line $C_1$ from $(0, 0)$ to $(x_1, 0)$ and the vertical line from $(x_1, 0)$ to $(x_1, y_1)$, $C = C_1 + C_2$.

We have

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_{C_1} \vec{F} \cdot \mathrm{d}\vec{r} + \int_{C_2} \vec{F} \cdot \mathrm{d}\vec{r}.$$

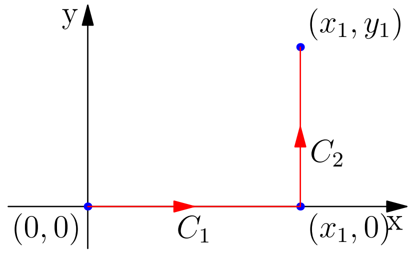


FIGURE 1. The curve $C$

Let $x(t) = t$, $y(y) = 0$, a parametrisation of $C_1$. Then

$$\vec{F} = 4t^2\hat{\imath} + 4t^2\hat{\jmath} \qquad \text{and} \qquad \mathrm{d}\vec{r} = \langle 1, 0\rangle\,\mathrm{d}t.$$

So

$$\int_{C_1} \vec{F}\cdot \mathrm{d}\vec{r} = \int_0^{x_1} \langle 4t^2, 4t^2\rangle \cdot \langle 1, 0\rangle\,\mathrm{dt} = \int_0^{x_1} 4t^2\,\mathrm{dt} = \Big[4t^3/3\Big]_0^{x_1} = 4x_1^3/3.$$

Let $x(t) = x_1$, $y(y) = t$, a parametrisation of $C_2$. Then

$$\vec{F} = (4x_1^2 + 8x_1 t)\hat{\imath} + (3t^2 + 4x_1^2)\hat{\jmath} \qquad \text{and} \qquad \mathrm{d}\vec{r} = \langle 0, 1\rangle\,\mathrm{d}t.$$

So

$$\int_{C_2} \vec{F}\cdot\mathrm{d}\vec{r} = \int_0^{y_1} \langle 4x_1^2 + 8x_1 t, 3t^2 + 4x_1^2\rangle\cdot\langle 0, 1\rangle\,\mathrm{dt} = \int_0^{y_1} 3t^2 + 4x_1^2\,\mathrm{dt} = \Big[t^3 + 4x_1^2 t\Big]_0^{y_1} = y_1^3 + 4x_1^2 y_1.$$

So

$$f(x, y) = 4x^3/3 + y^3 + 4x^2 y + c,$$

where $c$ is a constant. Check

$$\nabla f = \langle 4x^2 + 8xy, 3y^2 + 4x^2\rangle = \vec{F},$$

as expected.

**Method 2:** We want to solve two PDE's

$$f_x = 4x^2 + 8xy \qquad \text{and} \qquad f_y = 3y^2 + 4x^2.$$

Now if we integrate the first equation with respect to $x$ we get

$$f(x, y) = \int f_x(x, y)\,\mathrm{d}x = 4x^3/3 + 4x^2 y + g(y),$$

where $g(y)$ is a function of $y$. The point here is that for every value of $y$, we get an integration constant. As we vary $y$ this integration constant can vary. Put differently, if we differentiate $g(y)$ with respect to $x$ then we get zero. So $f(x, y)$ is determined up to $g(y)$. Now plug this value for $f(x, y)$ into the second PDE.

$$4x^2 + \frac{dg}{dy} = 3y^2 + 4x^2.$$

Comparing we have

$$\frac{dg}{dy} = 3y^2.$$

Integrating with respect to $y$, we get

$$g(y) = y^3 + c,$$

where $c$ is an integration constant. So

$$f(x, y) = 4x^3/3 + 4x^2y + y^3 + c.$$

This is the same solution we got using the other method.

Let's introduce a quantity which measures how far the vector field $\vec{F}$ is from being conservative, the curl of $\vec{F}$,

$$\operatorname{curl} \vec{F} = N_x - M_y.$$

We have $\operatorname{curl} \vec{F} = 0$ if and only if $\vec{F}$ is a gradient field, if and only if $\vec{F}$ is conservative.

The curl of a vector field is a strange beast. If $\vec{F}$ is a velocity vector field, the curl is double the angular velocity of the rotation component of the motion.

**Example 21.4.** *If $\vec{F} = \langle a, b \rangle$ is a constant vector field, then $\operatorname{curl} \vec{F}$ is zero,*

*$\vec{F} = \langle x, y \rangle$ represents expanding motion, which has zero curl.*

*$\vec{F} = \langle -y, x \rangle$ represents rotation around the origin, the curl is $2$.*

If $\vec{F}$ is a force field then $\operatorname{curl} \vec{F}$ is the torque exerted on a test mass. This measures how much $\vec{F}$ imparts angular momentum. For translation motion, the force divided by the mass is the acceleration, the derivative of the velocity. For rotation, the torque divided by the moment of inertia is the angular acceleration, the derivative of the angular velocity.