

24. SIMPLY CONNECTED REGIONS; TRIPLE INTEGRALS

24.1. **Simply connected regions.** Recall the example of the vector field

$$\vec{F} = \frac{-y\hat{i} + x\hat{j}}{x^2 + y^2}.$$

Its curl is zero but this vector field is not conservative. If one looks at the unit circle C centred at the origin, then the line integral of \vec{F} around C is 2π . The point is that Green's theorem does not apply, as \vec{F} is not defined on the whole of the region R enclosed by C ; \vec{F} is not defined at the origin.

Definition 24.1. A region R is *simply connected* if the interior of every closed curve C is entirely contained in R .

Informally speaking, the region R does not contain any holes.

Question 24.2. Which of the following regions are simply connected?

- (1) the whole plane \mathbb{R}^2 .
- (2) the plane minus the origin.
- (3) the unit disk.
- (4) the plane minus a line.
- (5) a comb.

Theorem 24.3. Suppose the region R is simply connected and \vec{F} is a vector field on R which is differentiable.

Then \vec{F} is a gradient vector field if and only if $M_y = N_x$.

The proof is the same as when R is the whole of \mathbb{R}^2 ; the same proof works by the very definition of simply connected.

Let's go back to the example above. Let's try to understand what is going on when the region is not simply connected. Start with an annulus, whose boundary is a circle of small radius plus a circle of arbitrary radius. Manufacture a closed curve by going around the big circle counterclockwise, going across a line segment to the small circle, going around the small circle, in the opposite direction and finally going back to the big circle, along the same line segment. Call the whole curve C . This encloses a region R on which \vec{F} is defined everywhere. With this choice of orientation of C , R is always on the left.

The curl is zero, so

$$\oint_C \vec{F} \, d\vec{r} = \iint_R \text{curl } F \, dA = 0.$$

But

$$\oint_C \vec{F} \, d\vec{r} = \int_{C_b} \vec{F} \, d\vec{r} + \int_L \vec{F} \, d\vec{r} - \oint_{C_s} \vec{F} \, d\vec{r} - \int_L \vec{F} \, d\vec{r},$$

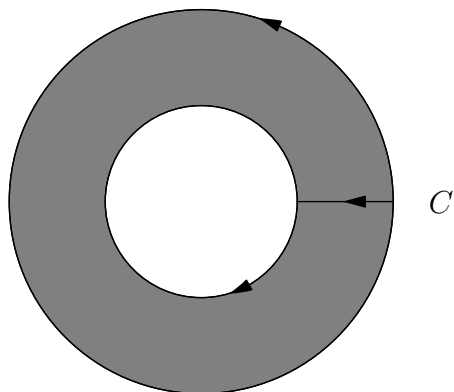


FIGURE 1. The region R with boundary C

where C_b is the big circle, C_s is the small circle, and L is the line segment.

Putting all of this together, we get

$$\int_{C_b} \vec{F} \, d\vec{r} = \oint_{C_s} \vec{F} \, d\vec{r}.$$

So the line integral is independent of the size of the circle; in fact any simple closed curve around the origin has integral 2π .

24.2. Triple integrals. Suppose we have a function $f(x, y, z)$ of three variables defined on a region R in space \mathbb{R}^3 . We can dice up R into small regions R_i and form the Riemann sum

$$\sum_i f(x_i, y_i, z_i) \Delta V_i,$$

where ΔV_i is the volume of R_i and (x_i, y_i, z_i) is a random point belonging to R_i . Letting ΔV_i go to zero we get the **triple integral**

$$\iiint_R f \, dV.$$

Example 24.4. What is the volume of the region R between the two paraboloids $z = x^2 + y^2$ and $z = 4 - x^2 - y^2$?

We want to calculate

$$\text{vol}(R) = \iiint_R 1 \, dV.$$

As usual imagine choosing V_i equal to boxes. First summing over z , then y and then x , the integral reduces to a triple integral

$$\iiint_R 1 \, dV = \int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} 1 \, dz \, dy \, dx,$$

where we need to determine the limits. The inner limit is the easiest. Given x and y , z ranges between $x^2 + y^2$ and $4 - x^2 - y^2$. To find the limits for x and y , consider the shadow of R in the xy -plane, the points in the xy -plane which are in the shade, if light comes vertically down.

We want to find those points (x, y) for which the point on the surface $z = 4 - x^2 - y^2$ is above the point on the surface $x^2 + y^2$, that is, we want those points (x, y) such that

$$x^2 + y^2 < 4 - x^2 - y^2.$$

Simplifying, we want

$$x^2 + y^2 < 2,$$

which is a disk of radius $\sqrt{2}$ centred at the origin. So we have

$$\iiint_R 1 \, dV = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{4-x^2-y^2} 1 \, dz \, dy \, dx,$$

We could have found the limits quicker by setting $x^2 + y^2 = 4 - x^2 - y^2$; however this won't always work. Now notice that this would come out much better if we used polar coordinates in the xy -plane.

$$\iiint_R 1 \, dV = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_{r^2}^{4-r^2} r \, dz \, dr \, d\theta,$$

Cylindrical coordinates (r, θ, z) are precisely the coordinates one gets this way. z is the usual height from the xy -plane, and r and θ are the usual polar coordinates of the projection of the point P down to the xy -plane. So r is the distance of P to the z -axis and θ is the angle the shortest line from P to the z -axis makes with the plane $y = 0$, the xz -plane. Of course

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \\ z &= z, \end{aligned}$$

gets one from cylindrical coordinates to the usual Cartesian coordinates. Note that the volume element

$$dV = dx \, dy \, dz = r \, dr \, d\theta \, dz.$$

As usual, we get this by considering what happens for a small change in the coordinates.

It is interesting to look at simple equations in cylindrical coordinates. For example, we already know

$$z = c,$$

c a constant, corresponds to a horizontal plane.

$$r = a,$$

a a constant, corresponds to a cylinder of radius a , centred along the z -axis. The equation

$$\theta = b,$$

b a constant, corresponds to a half plane, starting at the z -axis. Even simple equations look quite strange,

$$r = \theta,$$

corresponds to a scroll (infinite at both ends, parallel to the z -axis); if we take a cross-section, we get a spiral. The equation

$$z = ar,$$

represents a cone, vertex at the origin, central axis the z -axis.

Triple integrals have the same sort of applications as double integrals. If R is composed of material, with density δ , the mass of R is the integral,

$$\iiint_R \delta \, dV.$$

If $f(x, y, z)$ is a function on R , the average value \bar{f} of $f(x, y, z)$ is

$$\bar{f} = \frac{1}{V} \iiint_R f \, dV.$$

Here V is the volume of R . In particular, the centre of mass, with coordinates $(\bar{x}, \bar{y}, \bar{z})$ is given by

$$\bar{x} = \frac{1}{V} \iiint_R x \, dV \quad \bar{y} = \frac{1}{V} \iiint_R y \, dV \quad \text{and} \quad \bar{z} = \frac{1}{V} \iiint_R z \, dV.$$

Note that in the example above, $\bar{x} = \bar{y} = 0$ by symmetry.

We can also calculate the moment of inertia about an axis;

$$\iiint_R d^2 \delta \, dV,$$

where δ is the mass density and d^2 is the square of the distance to the axis. For the z -axis, we get

$$I_z = \iiint_R (x^2 + y^2) \delta \, dV = \iiint_R r^2 \delta \, dV,$$

which is consistent with the story in the plane. For the x -axis and the y -axis, we get

$$I_x = \iiint_R (y^2 + z^2) \delta \, dV \quad \text{and} \quad I_y = \iiint_R (x^2 + z^2) \delta \, dV.$$

Note that is consistent with the story in the plane, setting $z = 0$.

Example 24.5. *What is the moment of inertia of the solid cone between $z = ar$ and $z = b$ about the z -axis?*

We have

$$I_z = \iiint_R r^2 \, dV = \int_0^b \int_0^{2\pi} \int_0^{z/a} r^3 \, dr \, d\theta \, dz.$$

The limits for θ are easy, using the symmetry about the z -axis. We want to express r as a function of z . To find the upper and lower limits for r , note that we can squish the cone down to a plane (using the symmetry in θ), say the xz -plane.

In this case, the distance r to the z -axis simplifies to x , that is, $r = x$, when $y = 0$. We get a triangle in the xz -plane. We only take the half of the triangle $x \geq 0$, since take $0 \leq \theta \leq 2\pi$. Equivalently, the solid cone is a solid of revolution. Revolve the triangle in the xz -plane around the z -axis.

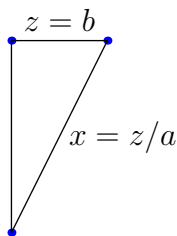


FIGURE 2. Half of squished cone

So $0 \leq r = x \leq z/a$.

Example 24.6. *What is the volume of the region where $z > 1 - y$ and $x^2 + y^2 + z^2 < 1$?*

The bottom surface is the plane $z = 1 - y$ and the top surface is the upper hemisphere $z = \sqrt{1 - x^2 - y^2}$ of the unit sphere. The intersection is a circle. The shadow of R is formed by the image of this circle, which will be an ellipse.

The volume is therefore

$$\iiint_R 1 \, dV = \int_0^1 \int_{-\sqrt{2y(1-y)}}^{\sqrt{2y(1-y)}} \int_{1-y}^{\sqrt{1-x^2-y^2}} 1 \, dz \, dx \, dy.$$

The shadow is the region

$$1 - y < \sqrt{1 - x^2 - y^2}.$$

Squaring both sides, we want

$$1 - 2y + y^2 < 1 - x^2 - y^2.$$

That is

$$x^2 < 2y(1 - y),$$

so that

$$-\sqrt{2y(1-y)} < x < \sqrt{2y(1-y)}.$$

Since $x^2 \geq 0$, we must have $2y(1-y) > 0$. So $0 < y < 1$.