26. Spherical coordinates; applications to gravitation

We have already seen that sometimes it is better to work in cylindrical coordinates. Spherical coordinates (ρ, ϕ, θ) are like cylindrical coordinates, only more so. ρ is the distance to the origin; ϕ is the angle from the z-axis; θ is the same as in cylindrical coordinates.

To get from spherical to cylindrical, use the formulae:

$$\begin{aligned} r &= \rho \sin \phi \\ \theta &= \theta \\ z &= \rho \cos \phi. \end{aligned}$$

As

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \\ z &= z, \end{aligned}$$

we have

$$\begin{aligned} x &= \rho \cos \theta \sin \phi \\ y &= \rho \sin \theta \sin \phi \\ z &= \rho \cos \phi. \end{aligned}$$

On the other hand,

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}.$$

The equation

$$\rho = a,$$

represents the surface of a sphere. On the surface of the sphere, ϕ constant corresponds to *latitude*, although $\phi = 0$ represents the north pole, $\phi = \pi/2$ represents the equator and $\phi = \pi$ represents the south pole. θ constant represents *longitude*.

Question 26.1. *What does the equation*

$$\phi = \pi/4$$

represent?

It represents a cone, through the origin. In cylindrical coordinates we have

$$z = r = \sqrt{x^2 + y^2}.$$

On the other hand, the equation

$$\phi = \pi/2,$$

represents the xy-plane.

We already know the volume element in Cartesian and cylindrical coordinates:

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z.$$

How about in spherical coordinates? We have to calculate the volume of the region when we have a small change in all three coordinates, $\Delta\rho$, $\Delta\theta$ and $\Delta\phi$.

First what happens if we take a sphere of constant radius $\rho = a$? $\Delta\theta$ and $\Delta\phi$ trace out a small region on the surface of the sphere, which is approximately a rectangle. The side corresponding to $\Delta\phi$ is part of the arc of a great circle of radius a. So the length of this side is $a\Delta\phi$. The side corresponding to $\Delta\theta$ is part of the arc of a circle, of radius $r = a\sin\phi$. So the length of this side is $a\sin\phi\Delta\theta$. The area of the region is therefore approximately

$$a^2\sin\phi\Delta\theta\Delta\phi.$$

The volume is then approximately given by

$$\Delta V \approx \rho^2\sin\phi\Delta\theta\Delta\phi\Delta\rho.$$

So

$$\mathrm{d}V = \rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta.$$

Let's consider again:

Example 26.2. *What is the volume of the region where $z > 1 - y$ and $x^2 + y^2 + z^2 < 1$?*

Note that the closest point on the plane $z = 1 - y$ to the origin is $(1/2, 1/2)$. So the distance of the plane $z = 1 - y$ from the origin is $1/\sqrt{2}$. If we rotate the plane so it is horizontal, we want the volume of the region above the horizontal plane

$$z = \frac{1}{\sqrt{2}},$$

inside the sphere. We can figure this out in cylindrical or spherical coordinates. We carry out the caculation in spherical coordinates for practice.

The plane is given by

$$\rho\cos\phi = z = \frac{1}{\sqrt{2}} \qquad \text{that is} \qquad \rho = \frac{\sec\phi}{\sqrt{2}}.$$

The region is symmetric with respect to θ, so that

$$0 \leq \theta \leq 2\pi.$$

For ϕ we start at the North pole and we go down to $\pi/4$. So the volume is

$$\int_0^{2\pi} \int_0^{\pi/4} \int_{\frac{1}{\sqrt{2}} \sec\phi}^{1} \rho^2 \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\theta.$$

The force due to gravity on a point mass m at the origin by a body of mass ΔM at (x, y, z) is given by

$$|\vec{F}| = \frac{Gm\Delta M}{\rho^2}.$$

Thus

$$\vec{F} = \frac{Gm\Delta M}{\rho^3}\langle x, y, z\rangle.$$

If we have a body, with mass density δ, then we have to sum together the contributions from each little piece of mass $\Delta M \approx \delta \Delta V$. Thus the force due to gravity on a point mass at the origin is

$$\vec{F} = \iiint_R \frac{Gm\langle x, y, z\rangle}{\rho^3} \delta \, \mathrm{d}V.$$

So the z-component of the force is

$$F_z = \iiint_R \frac{Gmz}{\rho^3} \delta \, \mathrm{d}V.$$

In general, always try to place the point mass at the origin and put the body so that the z-axis is an axis of symmetry (if this is possible). Then

$$\vec{F} = \langle 0, 0, F_z\rangle,$$

and it suffices to compute the z-component. In spherical coordinates, we get

$$\begin{aligned}
F_z &= Gm \iiint_R \frac{z}{\rho^3} \delta \, \mathrm{d}V \\
&= Gm \iiint_R \frac{\rho \cos\phi}{\rho^3} \rho^2 \sin\phi \delta \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\theta \\
&= Gm \iiint_R \delta \cos\phi \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\theta.
\end{aligned}$$

Newton's Theorem To calculate the gravitational attraction of a spherical planet of uniform density, one may treat the sphere as a point mass.

Let's show this is true when the point mass is on the surface of the sphere. Assume the planet has radius a, put the point mass at the origin and make this the south pole of the sphere. Then

$$F_z = Gm \iiint_R \delta \cos\phi \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\theta$$
$$= Gm \int_0^{2\pi} \int_0^{\pi/2} \int_0^{2a\cos\phi} \delta \cos\phi \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\theta.$$

The inner integral is

$$\int_0^{2a\cos\phi} \delta \cos\phi \sin\phi \, \mathrm{d}\rho = \Big[\delta \cos\phi \sin\phi \rho\Big]_0^{2a\cos\phi} = 2a\delta \cos^2\phi \sin\phi.$$

The middle integral is

$$\int_0^{\pi/2} 2a\delta \cos^2\phi \sin\phi \mathrm{d}\phi = \Big[-\frac{2}{3} a\delta \cos^3\phi\Big]_0^{\pi/2} = \frac{2}{3} a\delta.$$

The outer integral is

$$\int_0^{2\pi} \frac{2}{3} a\delta \, \mathrm{d}\theta = \Big[\frac{2}{3} a\delta\Big]_0^{2\pi} = \frac{4\pi}{3} a\delta.$$

So the integral is

$$Gm \frac{4\pi}{3} a\delta = \frac{GmM}{a^2},$$

since the mass of the planet is

$$M = \delta \frac{4\pi a^3}{3}.$$