29. The divergence theorem

Theorem 29.1 (Divergence Theorem; Gauss, Ostrogradsky)**.** *Let S be a closed surface bounding a solid D, oriented outwards. Let $\vec{F}$ be a vector field with continuous partial derivatives. Then*

$$\iint_S \vec{F} \cdot \mathrm{d}\vec{S} = \iiint_D \nabla \cdot \vec{F}\, \mathrm{d}V.$$

Why is

$$\nabla \cdot \vec{F} = \operatorname{div} \vec{F} = P_x + Q_y + R_z$$

a measure of the amount of material created (or destroyed) at (x, y, z)? Well imagine a small box with one vertex at (x, y, z) and edges Δx, Δy and Δz. The flux through this box is the sum of the flux through the six sides. We can pair off opposite sides. Consider the sides parallel to the xy-plane, that is, orthogonal to the vector $\hat{k}$. Crossing the bottom side is approximately

$$R(x, y, z)\Delta x \Delta y,$$

Crossing the top side is approximately

$$R(x, y, z + \Delta z)\Delta x \Delta y.$$

By linear approximation,

$$R(x, y, z + \Delta z) \approx R(x, y, z) + R_z \Delta z,$$

and so the difference is approximately

$$R_z \Delta x \Delta y \Delta z.$$

By symmetry the contribution from the other two sides is approximately

$$P_x \Delta x \Delta y \Delta z \qquad \text{and} \qquad Q_y \Delta x \Delta y \Delta z.$$

Adding this together, we get

$$(P_x + Q_y + R_z)\Delta x \Delta y \Delta z,$$

which is approximately the amount of water being created or destroyed in the small box. Dividing through by

$$\Delta x \Delta y \Delta z$$

and taking the limit, we get the divergence.

Proof of (29.1). We argue as in the proof of Green's theorem. Firstly, we can prove three separate identities, one for each of P, Q and R. So we just need to prove

$$\iint_S \langle 0, 0, R \rangle \cdot \mathrm{d}\vec{S} = \iiint_D R_z\, \mathrm{d}V.$$

Now divide the region into small pieces, each of which is vertically simple, so that

$$a \leq x \leq b \qquad c \leq y \leq d \qquad \text{and} \qquad f(x,y) \leq z \leq g(x,y),$$

is the region lying over a rectangle in the xy-plane lying between the graph of two functions f and g.

It is enough, because of cancelling, to prove the result for such a region. We calculate both sides explicitly in this case. S has six sides; the four vertical sides and the top and bottom sides. The flux across the four vertical sides is zero, since $\vec{F}$ is moving up and down. The flux across the top side is

$$\begin{aligned}\iint_{S_{\text{top}}} \langle 0,0,R\rangle \cdot \mathrm{d}\vec{S} &= \iint_{S_{\text{top}}} \langle 0,0,R\rangle \cdot \langle -g_x, -g_y, 1\rangle \,\mathrm{d}x\,\mathrm{d}y \\ &= \int_c^d \int_a^b R(x,y,g(x,y))\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

The flux across the bottom side is similar, but it comes with the opposite sign, so the total flux is

$$\int_c^d \int_a^b R(x,y,g(x,y)) - R(x,y,f(x,y))\,\mathrm{d}x\,\mathrm{d}y.$$

For the RHS, we have a triple integral,

$$\iiint_D R_z \,\mathrm{d}V = \int_c^d \int_a^b \int_{f(x,y)}^{g(x,y)} R_z(x,y,z)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y.$$

The inner integral is

$$\int_{f(x,y)}^{g(x,y)} R_z(x,y,z)\,\mathrm{d}z = \Big[R(x,y,z)\Big]_{f(x,y)}^{g(x,y)} = R(x,y,g(x,y)) - R(x,y,f(x,y)),$$

so that both sides are indeed the same. $\square$

What can we say about a radially symmetric vector field $\vec{F}$, such that there is a single unit source at the origin and otherwise the divergence is zero? By radial symmetry, we have

$$\vec{F} = \frac{\langle x,y,z\rangle}{c},$$

for some c, to be determined. Consider the flux across a sphere S of radius a. The flux across S is

$$\iint_S \vec{F} \cdot \mathrm{d}\vec{S}.$$

If we orient S so that the unit normal points outwards, we have

$$\hat{n} = \frac{1}{a}\langle x, y, z\rangle \qquad \text{and} \qquad \vec{F} \cdot \hat{n} = \frac{a}{c}.$$

So the flux is

$$\frac{4\pi a^3}{c}.$$

Since the only source of water is at the origin and we are pumping in water at a rate of one unit there, we must have

$$\frac{4\pi a^3}{c} = 1 \qquad \text{so that} \qquad c = 4\pi a^3.$$

That is

$$\vec{F} = \frac{1}{4\pi}\frac{\langle x, y, z\rangle}{\rho^3}.$$

Typical examples of such force fields are gravity and electric charge.