Discrete Structures EPFL, Fall 2014

Solution to Graded Problem Set 4

Date: 10.10.2014 Due date: 17.10.2014

Problem 1. If B is finite then B is countable. So assume that B is infinite (and hence A is also infinite). Let $e : B \to A$ be defined as the map $e(x) = x$ for any $x \in B$. Clearly, e is injective. Hence, $|B| \le |A| = |\mathbb{N}_{\ge 0}|$, where the last equality follows from our assumption that A is countable and infinite. Let us now show that also $|\mathbb{N}_{\ge 0}| \le |B|$. This, together with $|B| \le |\mathbb{N}_{\ge 0}|$ will show that $|B| = |\mathbb{N}_{\ge 0}|$.

For this it suffices to find an injective map, call it $f : \mathbb{N}_{\ge 0} \to B$. Since by assumption A is countable and infinite there exists a bijection $g : \mathbb{N}_{\ge 0} \to A$.

We define the desired map f as follows. First define the injective map $h : \mathbb{N}_{\ge 0} \to \mathbb{N}_{\ge 0}$ as

$$h(i) = \min\{j > h(i-1) : g(j) \cap B \neq \emptyset\},$$

with $h(-1)$ defined as -1. Note that this map finds the consecutive indices of the elements of A which are in B. Now $f(i)$ is defined as $f(i) = g(h(i))$. Since h is injective and g is a bijection, it follows that f is injective.

Problem 2.

a) $2 + 4 + 8 + 16 + 32 + \cdots + 2^{28} = \sum_{i=1}^{28} 2^i = 2 \cdot \sum_{i=0}^{27} 2^i = 2 \cdot \frac{2^{28}-1}{2-1} = 2^{29} - 2.$

b) $112+113+114+\cdots+673 = \sum_{i=112}^{673} i = \sum_{i=1}^{562}(i+111) = \left(\sum_{i=1}^{562} i\right) + 111 \cdot 562 = \frac{562 \cdot 563}{2} + 62382 = 220585.$

c) Let $S = \sum_{i=0}^{k} ir^{i-1}$. Assume first that $r = 1$. In this case, $S = \sum_{i=0}^{k} i = \frac{k \cdot (k+1)}{2}$. Then, pick $r \neq 1$. In this case,

$$S = \sum_{i=1}^{k} ir^{i-1} = \sum_{i=1}^{k} r^{i-1} + \sum_{i=1}^{k}(i-1)r^{i-1} = \sum_{i=0}^{k-1} r^i + r \cdot \sum_{i=0}^{k-1} ir^{i-1}.$$

Since $\sum_{i=0}^{k-1} ir^{i-1} + kr^{k-1} = S$, we have that

$$S = \frac{r^k - 1}{r - 1} + r(S - kr^{k-1}).$$

By solving the last equation for S, we find

$$S = -\frac{r^k - 1}{(r-1)^2} - \frac{kr^k}{1-r}.$$

Note that if we take the limit of the last expression when $r \to 1$, we recover the initial result $S = \frac{k \cdot (k+1)}{2}$.

Problem 3. Let T be the equilateral triangle with side 1. Connect the midpoints of each side of T, so that T is divided into 4 smaller equilateral triangle regions each with side $1/2$. These 4 smaller regions will be our "pigeonholes". Now, if one selects 5 points (our "pigeons") inside T, then at least two of them will be contained inside the same smaller equilateral triangle by pigeonhole principle. Since the maximum distance between two points in one of these smaller triangles is the length of the side of that smaller triangle, namely $1/2$, we have showed that there exists a pair of points out of the group of five originally selected that will be at distance $\leq 1/2$ from one another.

Problem 4.

a) Consider the function $f(x) : \mathbb{R} \to (0, +\infty)$ defined as

$$f(x) = e^x.$$

It is easy to check that f is a bijection. Hence, $|(0, +\infty)| = |\mathbb{R}|$.

b) Consider the function $f(x) : [a, b] \to [c, d]$ defined as

$$f(x) = \frac{d-c}{b-a}x + \frac{ad-bc}{a-b}.$$

It is easy to check that f is a bijection. Hence, $|[a, b]| = |[c, d]|$.

c) Consider the function $f(x) : (0, 1) \to (1, +\infty)$ defined as

$$f(x) = \frac{1}{x}.$$

It is easy to check that f is a bijection. Hence, $|(0, 1)| = |(1, +\infty)|$.

Problem 5. We can list the elements of L^* as follows. Start with the empty string. Continue by listing the strings of length 1 in alphabetic order. Then, list the strings of length 2 in alphabetic order, then the strings of length 3, and so on. More formally, we consider the following mapping $f : L^* \to \mathbb{N}$:

- $f(\varnothing) = 1$.
- $f(a) = 1 + 1, \cdots, f(z) = 1 + 26$.
- $f(aa) = 1 + 26 + 1, \cdots, f(zz) = 1 + 26 + 26^2$.
- $f(aaa) = 1 + 26 + 26^2 + 1, \cdots, f(zzz) = 1 + 26 + 26^2 + 26^3$.

$\vdots$

In general, the strings of length i will be mapped in alphabetic order into the integers between $\sum_{j=0}^{i-1} 26^j + 1$ and $\sum_{j=0}^{i-1} 26^j + 26^i$. This mapping is clearly injective and, therefore, $|L^*| \leq |\mathbb{N}|$. The other inequality $|L^*| \geq |\mathbb{N}|$ is obvious. As a result, we have that $|L^*| = |\mathbb{N}|$.

Problem 6. Let A_k be defined as the set of roots of the polynomials of the form $c_0 + c_1 x + \cdots + c_k x^k$ with $c_k \in \mathbb{Z}$ and $|c_k| \leq k$. Each of these polynomial has at most k distinct real roots (by the fundamental theorem of algebra) and two polynomials with the same coefficients have the same roots. The number of distinct ways in which we can choose the coefficients of such polynomials is $(2k+1)^{k+1}$. Therefore, $|A_k| \leq k \cdot (2k+1)^{k+1}$, which implies that A_k is countable.

By definition of algebraic number, $\mathcal{A} = \bigcup_{k \in \mathbb{N}} A_k$. Since the countable union of countable sets is countable, $\mathcal{A}$ is a countable set.