Discrete Structures EPFL, Fall 2014

Solution to Problem Set 5

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Problem 1.

a) $\pi n^4 + 10n^3 + 10^{10}n^2 + 10^{10^{10}}n = (\pi + \frac{10}{n} + \frac{10^{10}}{n^2} + \frac{10^{10^{10}}}{n^3})n^4 \leq 7n^4$ for $n \geq k := \lceil \sqrt[3]{10^{10^{10}}} \rceil$ (since for such n's all the three fractions inside the parenthesis are less than 1).

b) $1 \cdot 2 + 2 \cdot 3 + \cdots + (n-1) \cdot n \leq (n-1) \cdot n + (n-1) \cdot n + \cdots + (n-1) \cdot n = (n-1)^2 \cdot n \leq n^3$.

c) $1^{2014} + 2^{2014} + \cdots + n^{2014} \leq n^{2014} + n^{2014} + \cdots + n^{2014} = n^{2015}$.

d) Using the identities $\lfloor x \rfloor \leq x$ and $\lceil x \rceil \leq x + 1$ we can write,

$$\begin{aligned}
\lceil n + \sqrt{7} \rceil \cdot \lfloor n^2 - \sqrt{2} \rfloor + \lceil n^4 + \sqrt{3} \rceil &\leq (n + \sqrt{7} + 1)(n^2 - \sqrt{2}) + (n^4 + \sqrt{3}) \\
&= n^4 + n^3 + (\sqrt{7} + 1)n^2 - \sqrt{2}n + (\sqrt{3} - \sqrt{2} - \sqrt{7}\sqrt{2}) \\
&= \left(1 + \frac{1}{n} + \frac{\sqrt{7} + 1}{n^2} - \frac{\sqrt{2}}{n^3} + \frac{\sqrt{3} - \sqrt{2} - \sqrt{7}\sqrt{2}}{n^4}\right) \cdot n^4 \\
&\leq 11n^4, \qquad \text{for } n \geq 1.
\end{aligned}$$

Problem 2. It is clear that the first algorithm uses fewer operations as n grows. In fact it can easily checked that $\lim_{n \to +\infty} \frac{n\sqrt{n}}{n^2 \log n} = \lim_{n \to +\infty} \frac{1}{\sqrt{n} \log n} = 0$.

Problem 3. Each execution of line 4 involves 2 additions. Consequently each round of the inner **for** loop requires $2i$ additions. Thus, the total number of additions performed in the algorithm is

$$\sum_{i=1}^{n^2} 2i = 2 \times \frac{1}{2} n^2(n^2 + 1) = n^2(n^2 + 1).$$

Furthermore $n^2(n^2 + 1) = \Theta(n^4)$ because,

$$n^4 \leq n^2(n^2 + 1) \leq 2n^4$$

for all $n \geq 1$.

Problem 4.

a) We need $n - 1$ additions to compute the sum of the elements of a length n vectors and $n - 1 = \Theta(n)$.

b) For multiplying each row by the vector, we need n multiplications and $n-1$ additions. Thus, in total we need n^2 multiplications and $n(n-1)$ additions which means in total $2n^2 - n$ operations. $2n^2 - n = \Theta(n^2)$.

c) We need to repeat the task of b) n times. Hence we need $n^3 - n^2 = \Theta(n^3)$ operations in total.

d) The result is a $n \times n$ matrix. To compute each element of the result, we need $\lceil\sqrt{n}\rceil$ multiplications and $\lceil\sqrt{n}\rceil - 1$ additions. Thus, in total, we need $n^2(2\lceil\sqrt{n}\rceil - 1) = \Theta(n^2\sqrt{n})$ operations.

Problem 5.

a) True. $\frac{1}{n^2} \geq \frac{1}{n^2}$ for $n \geq 1$.

b) False. $\lim_{n\to+\infty} \frac{1/n^a}{1/n^b} = n^{b-a} = +\infty$ (since $b > a$).

c) True.

$$\begin{aligned}\log(1+n) &\leq \log(2n) = 2\log(\sqrt{2n}) \\ &\leq 2\sqrt{2n} = 2\sqrt{2}\cdot\sqrt{n}\end{aligned}$$

for $n \geq 1$.

d) True. $\lim_{n\to+\infty} \frac{n^4 4^n}{\frac{1}{n^5}5^n} = \lim_{n\to+\infty} n^9 \left(\frac{4}{5}\right)^n = 0$.

e) False. $\lim_{n\to+\infty} \frac{2^{n^2}}{4^n 3^{n\log n}} = \lim_{n\to+\infty} 2^{n^2-2n-\log_2(3)n\log n} = +\infty$. This means for every $C \geq 0$, there exists $n_0 = n_0(M)$ such that for $n \geq n_0$, $\frac{2^{n^2}}{4^n 3^{n\log n}} \geq C$. That is, $2^{n^2} \geq C4^n 3^{n\log n}$ for $n \geq n_0(C)$. Thus, it is impossible to find k and C such that $2^{n^2} \leq C4^n 3^{n\log n}$ for $n \geq k$ (since for $n \geq \max\{k, n_0(C)\}$, $2^{n^2} \geq C4^n3^{n\log n}$).

f) True. For $n \geq 4$,

$$\begin{aligned}\frac{n!}{2^n} &= \frac{n\times(n-1)\times\cdots\times 4\times 3\times 2\times 1}{2^n} \\ &\geq \frac{4\times 4\times\cdots\times 4\times 3\times 2\times 1}{2^n} = \frac{4^{n-3}\times 6}{2^n} \\ &= \frac{4^n}{2^n}\times\frac{6}{64} = \frac{6}{64}2^n.\end{aligned}$$

g) True. For $n \geq 2$,

$$\begin{aligned}(n!!)^2 &= [n\times(n-2)\times(n-4)\times\cdots\times 2]\times[n\times(n-2)\times(n-4)\times\cdots\times 2] \\ &\geq [n\times(n-2)\times(n-4)\cdots\times 2]\times[(n-1)\times(n-3)\times(n-5)\times\cdots\times 1] \\ &= n!\end{aligned}$$

Problem 6.

- $100n^3 + n^2$ and $n^2 + n^3$ have the same order.
- $3n^3 + 2^n$ and $n^2 + 2^n$ have the same order.