Discrete Structures EPFL, Fall 2014

Graded Problem Set 4

Date: 10.10.2014 Due date: 17.10.2014

Instructions: Please write the solution for each problem on a separate piece of paper. Sort these pieces of paper, with Problem 1 on top. Add this page as cover page, fill in your name, Sciper number, and the list of collaborators, and staple them all together on the left top.

Rules: You are allowed and encouraged to discuss these problems with your colleagues. However, each of you has to write down her solution in her own words. If you collaborated on a homework, write down the name of your collaborators and your sources in the space below. No points will be deducted for collaborations. But if we find similarities in solutions beyond the listed collaborations we will consider it as cheating. Please note that EPFL has a VERY strict policy on cheating and you might be in BIG trouble. It is simply not worth it.

Grading: You will get 10 bonus points if you solve the last problem (i.e., Problem 6).

Collaborators and sources:

Name : ....

Sciper : ....

Problem 1 ____________________ ... / 20

Problem 2 ____________________ ... / 30

Problem 3 ____________________ ... / 20

Problem 4 ____________________ ... / 30

Problem 5 ____________________ ... / 20

Problem 6 ____________________ + ... / 10

TOTAL ____________________ ... / 120

Problem 1. Let A be a countable set and let $B \subseteq A$. Prove that B is countable.

Problem 2. Compute the following sums:

a) $2 + 4 + 8 + 16 + 32 + \cdots + 2^{28}$

b) $112 + 113 + 114 + \cdots + 673$

c) $\sum_{i=0}^{k} i r^{i-1}$ for any $r \in \mathbb{R}$ and any $k \in \mathbb{N}$.

Problem 3. Prove that for any five points selected inside an equilateral triangle with side equal to 1, there always exists a pair whose distance is $\leq 1/2$.

Problem 4. Prove the following equalities between set cardinalities by explicitly showing an appropriate mapping.

a) $|(0, +\infty)| = |\mathbb{R}|$.

b) $|[a, b]| = |[c, d]|$, where $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$

c) $|(0, 1)| = |(1, +\infty)|$.

Problem 5. Let $L = \{a, b, c, \ldots, z\}$ denote the set of all lower-case letters and L^* denote the set of all strings consisted of the letters in L (this includes the null string). Show that

$$|L^*| = |\mathbb{N}|.$$

Problem 6. (Bonus) We call a real number x *algebraic* if x is the root of a polynomial equation $c_0 + c_1 x + \cdots + c_n x^n$ where all c_i's are integers. For example $\pm\sqrt{3}$ are algebraic numbers since they are roots of the polynomial $x^2 - 3$. Let $\mathcal{A}$ denote the set of algebraic numbers. Show $\mathcal{A}$ is a countable set.

Hint. We know that the countable union of countable sets is countable.