Discrete Structures EPFL, Fall 2014

Problem Set 9

Date: 14.11.2014 Not graded

Problem 1. Let f_n be the n-th element of the Fibonacci sequence which is recursively defined as

$$\begin{cases} f_0 = 0 \\ f_1 = 1 \\ f_{n+1} = f_n + f_{n-1} \quad \text{for } n \geq 1 \end{cases}$$

Prove the following equalities.

(a) $\sum_{i=1}^{n} f_i^2 = f_n \cdot f_{n+1}$ for any $n \geq 1$.

(b) $f_{n+1} \cdot f_{n-1} - f_n^2 = (-1)^n$ for any $n \geq 1$.

(c) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Then $A^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}$ for any $n \geq 1$.

Problem 2. Consider the following algorithm:

Algorithm 1 `LoopRec`

Require: n: positive integer

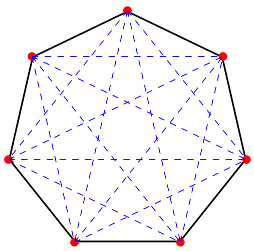
```
for i = 1, ..., n do
    print "Counting is a lot of fun"
if n > 1 then
    LoopRec(⌊n/3⌋)
```

How many times is the phrase "Counting is a lot of fun" printed in **Algorithm 1**? To make your life easier, only look at the case where n is a power of 3.

Problem 3. Prove that the number of diagonals of a regular convex n-gon is equal to

$$\frac{n(n-3)}{2}.$$

Remark. A regular convex n-gon is a convex polygon with n vertices s.t. all angles are equal in measure and all sides have the same length. A diagonal is a line connecting two non-adjacent vertices:



A regular convex 7-gon with its 14 diagonals (dashed).

Problem 4. Consider all strings of length 8 made up of the 26 lower-case letters $\{\mathtt{a}, \mathtt{b}, \ldots, \mathtt{z}\}$.

(a) How many begin with `the`?

(b) How many begin with `ab` *and* end with `yz`?

(c) How many begin with `ab` *or* end with `yz`?

(d) How many have exactly four `q`'s?

(e) (*Bonus.*) Consider all binary strings of length 12, i.e., made up of 0's and 1's. How many have exactly four 1's? How many have exactly four 1's, none of which are adjacent to each other?