Discrete Structures EPFL, Fall 2014

Problem Set 10

Date: 21.11.2014 Not graded

Problem 1. A professor teaching a *Discrete Structures* course gives a multiple choice quiz that has ten questions, each with four possible responses: `a`, `b`, `c`, `d`. What is the minimum number of students that must be in the professor's class in order to guarantee that at least three answer sheets must be identical? (Assume that no answers are left blank.)

Problem 2. Here is an incorrect solution to a problem. Find the error, explain why it is not correct, and give the correct answer.

Problem: Consider a standard deck of 52 cards, in which half of them are red ("diamonds" and "hearts") and the other half is black ("clubs" and "spades"). How many 5 card hands can be made with at least three red cards?

Solution: First, there are $\binom{26}{3}$ ways of choosing three red cards among the total of 26 red cards. Since the remaining two cards can be black or red, we can choose them from any of the 49 unused cards, i.e., there are $\binom{49}{2}$ distinct ways of choosing them. As a result, the final answer is $\binom{26}{3} \cdot \binom{49}{2}$.

Problem 3. The aim of this exercise is to prove the following result:

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n, \qquad n \in \mathbb{N}. \tag{1}$$

(a) Denote by E the set of binary strings of length n and by E_i the set of strings which contain exactly i 1's for $i \in \{0, 1, \cdots, n\}$. Convince yourself that $E = E_0 \cup E_1 \cup \cdots \cup E_n$.

(b) Compute $|E_i|$ for all $i \in \{0, 1, \cdots, n\}$.

(c) Compute $|E_i \cap E_j|$ for all $i, j \in \{0, 1, \cdots, n\}$ with $i \neq j$.

(d) Conclude evaluating $|E|$ directly and with the inclusion-exclusion principle.

Problem 4. In this problem, we develop a second proof of the result (1).

(a) First of all, by using the definition of binomial coefficient, prove that for any $m \geq 1$ and any $1 \leq n \leq m-1$,

$$\binom{m}{n} = \binom{m-1}{n} + \binom{m-1}{n-1}. \tag{2}$$

(b) Then, prove (2) with a combinatorial argument using the hint below.

Hint. Suppose you have m white balls. The binomial coefficient $\binom{m}{n}$ counts, e.g., the number of ways in which you can color n of them in black. Now, choose one specific ball. Either you decide to color this ball black or you leave it white. In how many ways can you color the remaining $m-1$ balls in each of these two cases?

(c) Use formula (2) to prove (1) by the Principle of Mathematical Induction.

Problem 5. Suppose inflation continues at three percent annually. That is, an item that costs 1 CHF now will cost 1.03 CHF next year. Let a_n denote the value (or more precisely the *purchasing power*) of one Swiss franc after n years.

(a) Find a recurrence relation for a_n.

(b) What is the value of 1 CHF after 20 years?

(c) After how many years the purchasing power is ≤ 0.1?

(d) Would it be better to suffer from an inflation of three percent annually for 80 years or from an inflation of ten percent annually for 20 years?

Problem 6. Consider the recurrence relation

$$a_n = 4a_{n-1} - 3a_{n-2}. \tag{3}$$

(a) Show that

$$a_n = \alpha 3^n + \beta$$

is a solution of the recursion (where α and β are arbitrary real numbers).

(b) Assume we additionally specify the initial conditions

$$a_0 = \pi \qquad \text{and} \qquad a_1 = \pi + 3\sqrt{2}. \tag{4}$$

Find the solution of the recurrence.

(c) Solve the recurrence (3) with initial conditions (4) by using generating functions.