Discrete Structures EPFL, Fall 2014

Special Problem Set 14 – Final Exam Preparation

Date: 19.12.2014 Not graded

Additional training for the final exam. On the following pages you can find the final exam we gave last year.

Rules:

- This exam is closed book. No electronic items are allowed. Place all your personal items on the floor. Leave only a pen and your ID on the desk. If you need extra scratch paper, please ask for it by raising your hand.
- Please do not cheat. We will be forced to report any such occurrence to the president of EPFL. This is not how you want to meet him. :-(
- The exam starts at 4:15pm and lasts till 7:15pm.
- If a question is not completely clear to you, don't waste time and ask us for clarification right away.
- It is not necessarily expected that you solve all problems. Don't get stuck. Start with the problems which seem the easiest to you and try to collect as many points as you can.
- For each of the following multiple-choice questions there is exactly one correct answer. Mark your answer on the **answer sheet**. The answer sheet is the only thing we will grade. No points will be subtracted for wrong answers.
- You are also asked to provide four proofs. Write your proofs also on the **answer sheet**. One more time: only solutions on the answer sheet count. You can answer in **Français**, **Deutsch**, **English**, **Italiano**, and **Română**.

Règles:

- *Cet examen se déroulera à livre fermé. Aucun appareil électronique n'est autorisé. Déposez toutes vos affaires personnelles sur le sol. Gardez seulement un stylo et votre carte CAMIPRO sur le pupitre. Si vous avez besoin de feuille de brouillon, demandez-en en levant la main.*
- *S'il vous plait, ne trichez pas. Nous serions obligés de rapporter n'importe qu'elle infraction au président de l'EPFL. Ce n'est certainement pas de cette façon que vous souhaitez le rencontrer :-(*
- *L'examen débute à 16:15 précise et se termine à 19:15.*
- *Si une question n'est pas entièrement claire pour vous, ne perdez pas de temps et demandez-nous immédiatement une explication supplémentaire.*
- *Il n'est pas forcément attendu que vous résolviez tous les problèmes. Ne restez pas bloqués! Commencez par les problèmes qui vous paraissent les plus simples et essayez d'obtenir le plus de points possibles.*
- *Pour chaque question à choix multiples, il y a exactement une réponse correcte. Indiquez votre réponse sur la feuille de réponses. Seule, cette feuille de réponses sera notée. Aucun point ne sera soustrait pour les mauvaises réponses.*
- *Il vous sera aussi demandé de produire quatre preuves. Écrivez-les aussi sur la feuille de réponses uniquement. Une fois de plus : seules les réponses sur cette feuille seront prises en compte. Vous pouvez répondre en* **Français**, **Deutsch**, **English**, **Italiano**, *et* **Română**.

MULTIPLE-CHOICE QUESTIONS – mark your answers on the answer sheet [*QUESTIONS A CHOIX MULTIPLES – marquez vos réponses sur la feuille de réponses*]

1. For each of the following sets, mark the correct answer. [*Pour chacun des ensembles suivants, marquez la réponse correcte.*] **[5 x 3 pts]**

A. finite [*fini*] **B.** infinite and countable [*infini et dénombrable*] **C.** uncountable [*non dénombrable*]

i) $\{\emptyset\} \times \mathbb{R}$
ii) $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Z}$
iii) $\{m \in \mathbb{Z} : m^2 \equiv 2 \pmod 4\}$
iv) The set of all even primes [*L'ensemble de tous les nombres premiers pairs*]
v) $\{c \in \mathbb{R} : c = a\sqrt{2} + b\sqrt{3}$ for some $a, b \in \mathbb{Q}\}$ [$\{c \in \mathbb{R} : c = a\sqrt{2} + b\sqrt{3}$ *pour* $a, b \in \mathbb{Q}\}$]

Solution.

i) C. Has the same cardinality as $\mathbb{R}$ since there is a bijection to $\mathbb{R}$.

ii) B. The Cartesian product of countable sets is countable.

iii) A. In fact it is the empty set. There are no squares of the form $4k + 2$.

iv) A. There is only one even prime.

v) B. There is a bijection of this set to $\mathbb{Q}^2$.

2. Consider the recursion $g_n = g_{n-1}g_{n-2}$, $n \geq 2$, and $g_0 = 1$, $g_1 = 2$. What is the solution? [*Quelle est la solution de la récurrence suivante?* $g_n = g_{n-1}g_{n-2}$, $n \geq 2$ *et* $g_0 = 1$, $g_1 = 2$] **[5pts]**

A. $1 + \frac{3}{2}n - \frac{1}{2}n^2$

B. 2^{f_n}, where f_n is the Fibonacci sequence, with $f_0 = 0$ and $f_1 = 1$

C. $1 + \frac{5}{2}n - 2n^2 + \frac{1}{2}n^3$

D. $2^{\frac{3}{2}n - \frac{1}{2}n^2}$

E. 2^n

Solution. The correct answer is 2^{f_n}, where f_n is the Fibonacci sequence with $f_n = f_{n-1} = f_{n-2}$, $n \geq 2$, and $f_0 = 0$, $f_1 = 1$. To see this, take the log to the base 2 of the recursion of g_n and you get the standard Fibonacci sequence.

3. Suppose that you want to go from Milan to Siena. You take the train that goes from Milan to Florence. The train is scheduled to take 1 hour and 45 minutes for this trip but experiences a delay in minutes that can be modelled as a random variable that is uniform in $\{5, 10, 12, 15, 20, 40\}$. At the station, a friend picks you up and brings you to Siena by car. The trip should last 1 hour, but, due to intense traffic, there is a delay that is expressed in minutes and can be modelled as a random variable uniform in $\{10, 16, 25, 30, 40, 65\}$. What is the expected duration of the travel? [*Supposons que vous vouliez aller de Milan à Sienne. Vous prenez le train de Milan vers Florence. Le train doit prendre* 1 *heure et* 45 *minutes pour ce voyage mais a un retard qui est modélisé comme une variable aléatoire (en minutes) uniforme sur* $\{5, 10, 12, 15, 20, 40\}$. *Depuis la gare de Florence, un ami vous conduit jusqu'à Sienne en voiture. Le voyage devrait prendre 1 heure, mais la circulation cause un délai qui lui est modélisé comme une variable aléatoire (en minutes) uniforme sur* $\{10, 16, 25, 30, 40, 65\}$. *Quelle est l'espérance de la durée de voyage?*] **[5pts]**

A. 4 hours and 30 minutes

B. 2 hours and 57 minutes
C. 2 hours and 45 minutes
D. 3 hours and 33 minutes
E. 3 hours and 9 minutes

Solution. The correct answer is 3 hours and 33 minutes. Let X be the random variable that expresses the length of the train trip and let Y be the random variable that expresses the length of the car trip. We want to compute $\mathbb{E}[X+Y]$. Using linearity of expectation, we have (in minutes) $\mathbb{E}[X+Y] = \mathbb{E}[X]+\mathbb{E}[Y] = 105+(5+10+12+15+20+40)/6+60+(10+16+25+30+40+65)/6$.

4. You and a friend of yours are taking an exam. There is lots of material to study. There are twenty topics and you know that you will be asked 2 of those, chosen uniformly at random. Since you do not have sufficient time to study everything, you two decide to skip one of these topics, namely Bayes' law. The exam is the same for both of you but takes place at different times. Your friend takes the exam first. The probability that he passes the exam if there is a question dealing with Bayes' law is $1/10$ but the probability that he passes the exam if there is no question concerning Bayes' law is $1/2$. You hear that your friend fails the exam! Should you be worried? More precisely, what is the probability that you will be asked a question concerning Bayes' law? [*Vous et votre ami passez un examen. Il y a beaucoup de materiel à étudier. Il y a vingt sujets et vous savez que vous serez examinés sur deux d'entre eux, choisis uniformément au hasard. Puisque vous n'avez pas sufisamment de temps pour étudier, vous et votre ami décidez de laisser tomber un sujet, la loi de Bayes. L'examen est le même pour vous deux, mais a lieu a des moments différents. Votre ami passe l'examen en premier. La probabilité qu'il réussisse l'examen s'il y a une question sur la loi de Bayes est* $1/10$*, alors que la probabilité qu'il réussisse s'il n'y a pas de question concernant la loi de Bayes est* $1/2$*. Votre ami échoue l'examen! Devriez-vous de faire du soucis? Plus précisement, quelle est la probabilité qu'on vous demandera une question concernant la loi de Bayes?*] **[5pts]**

A. $\frac{7}{23}$
B. $\frac{1}{10}$
C. $\frac{2}{9}$
D. $\frac{1}{6}$
E. $-\frac{1}{3}$

Solution. The correct answer is $1/6$. Let S be a binary random variable, where 1 means that your friend passes and 0 that he fails and let B be a binary random variable where 1 means that the exam contains Bayes' law and 0 that it does not. We know that $P(B = 1) = 1/10$, $P(S = 0 \mid B = 1) = 9/10$, and $P(S = 0 \mid B = 0) = 0.5$. We are interested in $P(B = 1 \mid S = 0)$. By Bayes' law this is equal to

$$P(B = 1 \mid S = 0) = \frac{P(S = 0 \mid B = 1)P(B = 1)}{P(S = 0)} = \frac{\frac{9}{10}\frac{1}{10}}{\frac{9}{10}\frac{1}{10} + \frac{1}{2}\frac{9}{10}} = \frac{1}{6}.$$

5. Let $a \in \mathbb{R}$, $b \in \mathbb{N}$, and $c \in \mathbb{Q}$. For each of the following couple of sets, mark the correct answer. [*Soit* $a \in \mathbb{R}$, $b \in \mathbb{N}$ *et* $c \in \mathbb{Q}$. *Pour chacun des couples suivants, indiquez la réponse correcte.*] **[4 x 3 pts]**

A. $|P| = |Q|$ **B.** $|P| > |Q|$ **C.** $|P| < |Q|$

i) $P = \{(a, b, c) \mid f(n) = cn^2 + bn + a \text{ is } o(n)\}$
$Q = \{(a, b, c) \mid a - b + c = \sqrt{3}\}$

ii) $P = \{(a, b, c) \mid f(n) = an^2 + bn + c \text{ is } \Theta(n)\}$
$Q = \{(a, b, c) \mid a + b + c \in \mathbb{R}\}$

iii) $P = \{(a,b,c) \mid f(n) = an^2 + bn + c \text{ is } O(n^4)\}$
$Q = \{(a,b,c) \mid b = c = 1\}$

iv) $P = \{(a,b,c) \mid f(n) = an^2 + bn + c \text{ is } \Omega(2^n)\}$
$Q = \{(a,b,c) \mid a = c = 3, b \leq 4\}$

Solution.

i) If $f(n)$ is $o(n)$, then $c = b = 0$ and we can freely choose a. Hence, $|P| = |\mathbb{R}|$. In the set Q, we can freely choose b and c, but then a is fixed by the fact that $a - b + c = \sqrt{3}$. Notice that if we choose a and b, there might not exist a c s.t. $a - b + c = \sqrt{3}$ (and the same reasoning applies choosing a and c). Hence, $|Q| = |\mathbb{N} \times \mathbb{Q}| = |\mathbb{N}|$ and $|P| > |Q|$.

ii) If $f(n)$ is $\Theta(n)$, then $a = 0$ and we can freely choose b and c. Hence, $|P| = |\mathbb{N} \times \mathbb{Q}| = |\mathbb{N}|$. The set Q contains all the possible (a,b,c). Hence, $|Q| = |\mathbb{R}|$ and $|Q| > |P|$.

iii) If $f(n)$ is $O(n^4)$, we can freely choose a, b, and c. Hence, $|P| = |\mathbb{R} \times \mathbb{N} \times \mathbb{Q}| = |\mathbb{R}|$. In Q you can freely choose a, so $|Q| = |\mathbb{R}| = |P|$.

iv) There exists no $f(n)$ which is $\Omega(2^n)$. Therefore, $P = \emptyset$. On the other hand, $Q \neq \emptyset$ and, therefore, $|P| < |Q|$.

6. A random variable is ... [*Une variable aléatoire est ...*] **[4pts]**

A. the outcome of a random experiment.[*le résultat d'une experience aléatoire.*]
B. a variable that is random.[*une variable qui est aléatoire.*]
C. a variable that is specified by a random function.[*une variable qui est definie par une application aléatoire.*]
D. a map from the sample space to the reals.[*une application de l'espace échantillon vers les réels.*]

Solution. Repeat after me ... a random variable is a map from the sample space to the reals!

7. Consider the generating function $F(x) = \sum_{i=1}^{5} \frac{1}{(x-1/i)^i}$. Let $f(n)$ be the sequence of numbers corresponding to this generating function, i.e., $F(x) = \sum_{n \geq 0} f(n)x^n$. Which of the following statements corresponding to the growth rate of $f(n)$ is correct? [*Considez la fonction génératrice* $F(x) = \sum_{i=1}^{5} \frac{1}{(x-1/i)^i}$. *Soit* $f(n)$ *la suite de nombres correspondant à cette fonction génératrice, i.e.,* $F(x) = \sum_{n \geq 0} f(n)x^n$. *Quel est le taux de croissance de* $f(n)$*?*] **[5pts]**

A. $\Theta(5^n n^5)$
B. $\Theta(5^{-n} n^5)$
C. $\Theta(n^4)$
D. $\Theta(5^n n^4)$
E. $\Theta(1)$

Solution. The correct answer is $\Theta(5^n n^4)$. Note that $\frac{1}{x-1/i} = -\frac{i}{1-ix} = -i\sum_{n \geq 0} i^n x^n$. So the larger the i the larger the growth rate. Hence the term which dominates is the term $\frac{1}{(x-1/5)^5}$. Note that we have the 5-th power. This means that we have to take the derivative of this formula. In particular, if we have the 5-th power then we have to take 4 times the derivative, and each time we get a factor of order n out.

8. Given $\mathbf{a} = (a_0, a_1, \cdots, a_9)$ with $a_i \in \{0,1,2,3,4\}$ for all i, define $f(\mathbf{a}) = |\{i \mid a_i \equiv 0 \pmod 2\}|$ and $g(\mathbf{a}) = |\{i \mid a_i \not\equiv 0 \pmod 2\}|$. What is $|\{\mathbf{a} \mid f(\mathbf{a}) > g(\mathbf{a})\}|$? In words, how **[5pts]**

many sequences of length 10 with elements in $\{0,1,2,3,4\}$ are there that contain more even than odd numbers? [*Soit* $\mathbf{a} = (a_0, a_1, \cdots, a_9)$ *avec* $a_i \in \{0,1,2,3,4\}$ *pour tout* i. *Définissez* $f(\mathbf{a}) = |\{i \mid a_i \equiv 0 \pmod 2\}|$ *et* $g(\mathbf{a}) = |\{i \mid a_i \not\equiv 0 \pmod 2\}|$. *Que vaut* $|\{\mathbf{a} \mid f(\mathbf{a}) > g(\mathbf{a})\}|$? *En mots, combien de suites de longueur* 10 *avec éléments dans l'ensemble* $\{0,1,2,3,4\}$ *contiennent plus de nombres pairs qu'impairs?*]

A. $\sum_{k=4}^{10} \binom{10}{k-1} \cdot 3^k \cdot 2^{10-k}$

B. $\sum_{k=5}^{10} \binom{10}{k} \cdot 2^k \cdot 3^{10-k}$

C. $\sum_{k=6}^{10} \binom{10}{k} \cdot 3^k \cdot 2^{10-k}$

D. 0

E. $\sum_{k=7}^{10} \binom{10}{k-1} \cdot 4^k \cdot 1^{k-5}$

Solution. The correct answer is: $\sum_{k=6}^{10} \binom{10}{k} \cdot 3^k \cdot 2^{10-k}$. Out of the 10 positions you can have 6, 7, 8, 9, or 10 with even numbers. This is the sum over k. Then, choose the positions of these k entries. This is represented by the binomial $\binom{10}{k}$. Finally there are 3 even and 2 odd numbers to be chosen.

9. For each of the following propositions, mark on the answer sheet whether they are true (**T**) or false (**F**). [*Pour chaque des propositions suivantes, indiquez sur la feuille réponse si elles sont vraies (***T***) ou fausses (***F***)*] **[4 x 2pts]**

i) If f and g are functions from a set X to itself then $f \circ g = g \circ f$. [*Si* f *et* g *sont des applications d'un ensemble* X *vers lui-même, alors* $f \circ g = g \circ f$.]

ii) For any real number x, $\lceil x \rceil = \lfloor x \rfloor + 1$. [*Pour tout nombre réel* x, $\lceil x \rceil = \lfloor x \rfloor + 1$.]

iii) $\forall S((S \subseteq \mathbb{R} \wedge S \neq \emptyset) \rightarrow (\exists x \in S\ \forall y \in S\ (x \leq y)))$.

iv) This is the best course I have ever taken. [*Ceci est le meilleur cours que j'ai jamais prix.*]

Solution.

i) F, take $f(x) = 2x$ and $g(x) = x^2$, then $(f \circ g)(x) = 2x^2$ but $(g \circ f)(x) = 4x^2$.

ii) F, e.g., take x to be an integer.

iii) F, e.g., take S to be all of $\mathbb{R}$ or take S to be an (half-)open interval like $(0, 1]$.

iv) T, obviously! But since we are open minded, you will get full points even if you answered incorrectly! :-)

10. The remainder of the division of 7^{42} by 8 is equal to [*Le reste de la division de* 7^{42} *par* 8 *vaut*] **[5pts]**

A. 1
B. 3

C. 7
D. 5
E. 0

Solution. The correct answer is 1 because $7^2 \equiv 49 \equiv 1 \pmod 8$ and so $7^{42} = (7^2)^{21} \equiv 1^{21} \equiv 1 \pmod 8$.

11. Consider a random variable X which assumes only strictly positive integer values and s.t. $\mathbb{P}(X = n) = c \cdot \alpha^n$. For a given value $\alpha \in (0,1)$ find the value of c so that this forms a proper probability distribution. What is c? [*Soit X une variable aléatoire qui prend des valeurs strictement positives et telle que $\mathbb{P}(X = n) = c \cdot \alpha^n$. Pour une valeur fixée $\alpha \in (0,1)$ trouver la valeur de c telle que ceci forme une distribution de probabilité.*] **[5pts]**

A. $\dfrac{\alpha}{1-\alpha}$

B. 1

C. $1-\alpha$

D. α

E. $\dfrac{1-\alpha}{\alpha}$

Solution. The correct answer is $c = \frac{1-\alpha}{\alpha}$ since $1 = \sum_{n\geq 1} \mathbb{P}(X = n) = c \sum_{n\geq 1} \alpha^n = c\frac{\alpha}{1-\alpha}$.

12. Consider the random variable X defined in the previous question. Compute $\mathbb{E}[X]$. [*Soit X la variable aléatoire definie à la question précédente. Calculez $\mathbb{E}[X]$.*] **[4pts]**

A. -2

B. α

C. $\dfrac{1-\alpha}{\alpha}$

D. $\dfrac{1}{\alpha}$

E. $\dfrac{1}{1-\alpha}$

Solution. The correct answer is $\mathbb{E}[X] = \frac{1-\alpha}{\alpha} \sum_{n\geq 1} n\alpha^n = \frac{1-\alpha}{\alpha}\alpha \left(\frac{1}{1-\alpha}\right)' = \frac{1}{1-\alpha}$.

13. Let $f(n) = n(n+1)(n+2)$. Which of the following is correct? [*Soit $f(n) = n(n+1)(n+2)$. Lequel des énoncés suivants est correct?*] **[4pts]**

A. For all $n \in \mathbb{N}$, $f(n)$ is divisible by 6 [*Pour tous $n \in \mathbb{N}$, $f(n)$ est divisible par 6*]
B. For all $n \in \mathbb{N}$, $f(n)$ is divisible by 4 [*Pour tous $n \in \mathbb{N}$, $f(n)$ est divisible par 5*]
C. For all $n \in \mathbb{N}$, $f(n)$ is divisible by 5 [*Pour tous $n \in \mathbb{N}$, $f(n)$ est divisible par 4*]
D. None of the previous [*Aucune de ces réponses.*]

Solution. The correct answer is that "For all $n \in \mathbb{N}$, $f(n)$ is divisible by 6". Note that this expression always contains the factor two since you have n as well as $n+1$, one of which must be even. Further, if $n \equiv 0 \pmod 3$ then n is divisible by 3 and we are done. If $n \equiv 1 \pmod 3$

then $n+2$ is divisible by 3 and we are done. If $n \equiv 2 \pmod 3$ then $n+1$ is divisible by 3 and we are done. The other statements are incorrect.

14. A logical proposition is called *weird* if choosing its variables uniformly at random the probability that the proposition is true is $1/4$. How many *weird* logical propositions are there with 8 distinct variables? You should count propositions that are logically equivalent (i.e. have the same truth table) only once. [*Une proposition logique est dite* bizarre *si on choisissant uniformément au hazard la valeur de ses variables, la proposition est vraie avec probabilité* 1/4. *Combien de propositions logiques bizarres y a-t-il avec 8 variables distinctes? Vous devrez compter les propositions logiquement équivalentes (i.e. ayant la même table de vérité) qu'une seule fois.*] **[5pts]**

- **A.** $\dfrac{8!}{6!2!}$
- **B.** 2^{256}
- **C.** $\dfrac{256!}{64!192!}$
- **D.** $3 \cdot 2^{64}$
- **E.** $\dfrac{128!}{64!64!}$

Solution. The correct answer is $\dfrac{256!}{64!192!}$. If we have 8 variables then the truth table has size $2^8 = 256$. Choosing the truth values of the variables uniformly at random means that we pick one of these rows at random. The statement says that exactly $1/4$ of those should have truth value T and the remaining ones truth value F. Hence, out of the 256 rows, 64 need to have the truth value T and so the number of such truth tables is $\binom{256}{64}$.

15. You roll a pair of dodecahedral dice. These are dice with 12 equiprobable faces, numbered from 1 to 12. What is the probability that the sum of the numbers on the dice is strictly less than 12? [*Vous lancez une paire de dés dodécaèdraux. Ceux sont des dés à 12 faces équiprobables, numérotées de 1 à 12. Quelle est la probabilité que la somme des chiffres affichés soit strictement inférieure à 12?*] **[5pts]**

A. $\frac{\binom{12}{2}}{12}$ **B.** $\frac{\binom{12}{2}}{12^2}$ **C.** $\frac{\binom{11}{2}}{12^2}$ **D.** $\frac{1}{2}$ **E.** $\frac{\binom{11}{2}}{12}$

Solution. The correct answer is $\binom{11}{2}/144$. Arrange all possible outcomes as a square containing 12 points per row and 12 rows. Each of these outcomes has probability 12^{-2}. The outcomes we are interested in are the ones in a triangular region. There are $10+9+8+\cdots+1 = \binom{11}{2}$ of those.

16. [BONUS QUESTION]You are given the following two logical statements: [*On vous donne les deux propositions logiques suivantes:*] **[4pts]**

$$
\begin{aligned}
&1. \quad \forall x(A(x) \vee (B(x) \to C(x))),\\
&2. \quad \neg\forall y(A(y) \vee C(y)).
\end{aligned}
$$

Among the following, what can you conclude from 1 and 2? [*Parmis les énoncés logiques suivants, que peut-on conclure de 1 et 2?*]

- **A.** $\forall x \neg A(x)$
- **B.** $\neg\exists x C(x)$
- **C.** $\exists y C(y)$
- **D.** $\exists z \neg B(z)$

Solution. 2 is equivalent to $\exists y \neg A(y) \wedge \neg C(y)$. Instantiate this (by using a new symbol t) to obtain $\neg A(t)$ and $\neg C(t)$. Instantiate 1 by plugging t for x, obtaining $A(t) \vee (B(t) \rightarrow C(t))$. Now this together with $\neg A(t)$ implies $B(t) \rightarrow C(t)$, which together with $\neg C(t)$, by modus tollens gives $\neg B(t)$. By existential generalization, we have $\exists z \neg B(z)$.

PROBLEMS – write down the proofs on the answer sheet [*PROBLEMES – écrivez les démonstrations sur la feuille de réponse*]

17. In this problem we will revisit the *birthday* paradox. Assume that you have a set $\mathcal{P}$ of people and that this set has cardinality P. Assume further that there are N days in a year and that each person in $\mathcal{P}$ has his/her birthday uniformly distributed over N and that these events are independent. **[10pts]**

- What is the expected number of unordered pairs of people (i.e., {John, Mary} = {Mary, John}) who have their birthday on the same day of the year as a function of P and N?
- How should you choose P as a function of N so that the expected number of such pairs is exactly 1? You can think of P and N as large and give an approximate answer.

[*Pour ce problème nous revisiterons le paradoxe des anniversaires. Supposez donné un ensemble $\mathcal{P}$ d'individus, de cardinalité P. Supposez de plus qu'il y a N jours dans l'année et que chaque individu dans $\mathcal{P}$ a son anniversaire, uniformément distribué durant l'année et que ces évenement sont indépendants.*

- *Quelle est l'espérance du nombre de paires d'individus (non ordonnées, i.e., {John, Mary} = {Mary, John}) qui partagent le même anniversaire en fonction de P et N?*
- *Comment devrait-on choisir P en function de N pour que l'espérence du nombre de telles paires soit 1? Vous pouvez penser que P et N sont grands et donner une réponse approximative.]*

Solution. There are $\binom{P}{2}$ pairs and for each pair the probability that they share the same birthday is $1/N$. So the answer to the first question is $P(P-1)/(2N)$. For large P this is very close to $P^2/(2N)$. Hence, we should choose $P=\sqrt{2N}$. This is essentially the same answer we got when we looked at the actual probabilities.

18. Prove that for all integers $n \geq 1$, **[10pts]**

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}}.$$

[*Démontrez que pour tout entier $n \geq 1$,*

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}}.]$$

Solution. For $n=1$ we have in fact equality, and so the base case is correct. Assume now that the inequality is correct for some $n \geq 1$. Let us now consider the inequality for $n+1$. We need to prove that

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}\cdot\frac{2n+1}{2n+2} \leq \frac{1}{\sqrt{3n+4}}.$$

By induction hypothesis, we have that

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}}.$$

Hence, if we prove that

$$\frac{2n+1}{2n+2}\frac{1}{\sqrt{3n+1}} \leq \frac{1}{\sqrt{3n+4}},$$

we have proved the claim. The latter is equivalent to

$$(2n+1)^2(3n+4) \leq (2n+2)^2(3n+1) \leftrightarrow -n \leq 0.$$

19. Fed up with all these silly exam questions, you decide to take up bartending in Satellite. Little did you know, DS will be with you forever! Your first customer orders a glass of red Rivella. The second customer requests a glass of green Rivella. From then on, each new customer tastes **[10pts]**

what the previous two had ordered, and asks for a mix consisting of 2/3 of the previous persons drink, and 1/3 of the one before that. What is the fraction of red Rivella contained in the n-th drink and what is the limit of the fraction when n tends to infinity? [*Non pouvant plus de toutes ces ridicules questions d'examen, vous décidez de devenir barman à Satellite. Vous ne saviez cependant pas que DS vous suivra à jamais! Votre premier client commande un verre de Rivella rouge. Votre second client commande un verre de Rivella vert. Chaque client subséquent commande un mélange consistant en 2/3 du contenu du verre du dernier client et en 1/3 du contenu du verre de l'avant-dernier client. Quelle est la proportion de Rivella rouge contenue dans le n-ème verre et quelle est sa valeur asymptotique?*]

Solution. Let a_n be the fraction of red Rivella in the n-th drink. We have $a_1 = 1$, $a_2 = 0$ and $a_n = \frac{2}{3}a_{n-1} + \frac{1}{3}a_{n-2}$. Let $A(x)$ denote the corresponding generating function, $A(x) = \sum_{n\geq 1} a_n x^n$. Then we have

$$A(x) - a_1 x - a_2 x^2 = \frac{2}{3}x(A(x) - a_1 x) + \frac{1}{3}x^2 A(x),$$

so that

$$\begin{aligned} A(x) &= \frac{x - \frac{2}{3}x^2}{1 - \frac{2}{3}x - \frac{1}{3}x^2} = 2 + \frac{1}{4(1-x)} - \frac{27}{12(1+x/3)} \\ &= 2 + \frac{1}{4}\sum_{n\geq 0} x^n - \frac{27}{12}\sum_{n\geq 0}(-1)^n 3^{-n} x^n \\ &= \frac{1}{4}\sum_{n\geq 1} x^n - \frac{27}{12}\sum_{n\geq 1}(-1)^n 3^{-n} x^n. \end{aligned}$$

So, the asymptotic value is $\frac{1}{4}$. If you want to avoid the partial fraction you can directly try out a linar combination of the solutions corresponding to the two roots, namely 1 and $-\frac{1}{3}$, of the charateristic equation.

20. Prove or give a counterexample of the following statement. Given any 6 integers, there must be at least two whose difference is divisible by 5. [Hint: Pigeons are not only tasty but also smarter than they look!] [*Démontrez ou donnez un contre-exemple de l'énoncé suivant. Étant donnés 6 entiers, il doit y en avoir au moins deux dont la différence est divisible par 5. [Indice: Les pigeons ne sont pas seulement délicieux, mais aussi plus intelligents qu'ils en ont l'air.]*] **[10pts]**

Solution. Take the 6 integers and reduce them modulo 5. Then at least two of those must have the same modulus by the pigeonhole principle. Pick such a pair. Then these two will have a difference that is zero modulo 5.