Discrete Structures EPFL, Fall 2014

Midterm Exam – Question Sheet

Date: 11.11.2014 – 8:15-10:00am **Version A**

Rules:

- This exam is closed book. No electronic items are allowed. Place all your personal items on the floor. Leave only a pen and your ID on the desk. If you need extra scratch paper, please ask for it by raising your hand.
- Please do not cheat. We will be forced to report any such occurrence to the president of EPFL. This is not how you want to meet him. :-(
- The exam starts at 8:15am and lasts till 10am.
- If a question is not completely clear to you don't waste time and ask us for clarification right away.
- It is not necessarily expected that you solve all problems. Don't get stuck. Start with the problems which seem the easiest to you and try to collect as many points as you can.
- For each of the following multiple-choice questions there is exactly one correct answer. Mark your answer on the **answer sheet**. The answer sheet is the only thing we will grade. No points will be subtracted for wrong answers.
- You are also asked to provide several proofs. Write your proofs also on the **answer sheet**. One more time: **only solutions on the answer sheet count**. You can answer in **Français**, **Deutsch**, **English**, **Italiano**, and **فارسی** .

Règles:

- *Cet examen se déroulera à livre fermé. Aucun appareil électronique n'est autorisé. Déposez toutes vos affaires personnelles sur le sol. Gardez seulement un stylo et votre carte CAMIPRO sur le pupitre. Si vous avez besoin de feuille de brouillon, demandez-en en levant la main.*
- *S'il vous plait, ne trichez pas. Nous serions obligés de rapporter n'importe qu'elle infraction au président de l'EPFL. Ce n'est certainement pas de cette façon que vous souhaitez le rencontrer :-(*
- *L'examen débute à 08:15 précise et se termine à 10:00.*
- *Si une question n'est pas entièrement claire pour vous, ne perdez pas de temps et demandez-nous immédiatement une explication supplémentaire.*
- *Il n'est pas forcément attendu que vous résolviez tous les problèmes. Ne restez pas bloqués! Commencez par les problèmes qui vous paraissent les plus simples et essayez d'obtenir le plus de points possibles.*
- *Pour chaque question à choix multiples, il y a exactement une réponse correcte. Indiquez votre réponse sur la feuille de réponses. Seule, cette feuille de réponses sera notée. Aucun point ne sera soustrait pour les mauvaises réponses.*
- *Il vous sera aussi demandé de produire quelques preuves. Écrivez-les aussi sur la feuille de réponses uniquement. Une fois de plus :* **seules les réponses sur cette feuille seront prises en compte**. *Vous pouvez répondre en* **Français**, **Deutsch**, **English**, **Italiano**, *et* **فارسی** .

MULTIPLE-CHOICE QUESTIONS – mark your answers on the answer sheet [*QUESTIONS A CHOIX MULTIPLES – marquez vos réponses sur la feuille de réponses*]

1. Which of the following expressions is equivalent to $\neg(\forall n \in \mathbb{N}\, P(n) \to Q(n))$? [*Laquelle parmi les expressions suivantes est équivalente à* $\neg(\forall n \in \mathbb{N}\, P(n) \to Q(n))$*?*] **[4pts]**

A. $\forall n \in \mathbb{N}\, P(n) \vee \overline{Q}(n)$
B. $\forall n \in \mathbb{N}\, \overline{P}(n) \wedge Q(n)$
C. $\exists n \in \mathbb{N}\, P(n) \wedge \overline{Q}(n)$
D. $\exists n \in \mathbb{N}\, \overline{P}(n) \vee Q(n)$
E. None of the previous

Solution: C

2. Mark each of the following statements T (true) or F (false). [*Marquez chacun des énoncés suivants T (vrai) ou F (faux).*] **[4 x 2 pts]**

(i) $383492 \times 389595 \equiv 1 \pmod{99484875}$.

(ii) If p is a prime, then $p^{p-1} \equiv 1 \pmod{p}$.

(iii) $7^{23} \equiv 7 \pmod{23}$.

(iv) $7^{2014} \equiv 49 \pmod{50}$.

Solution: F; F; T; T

3. Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a bijective function. Let $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be defined as below. Which of the following statements is *always* true, regardless of the specific choice of f? [*Soit $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ une application bijective. Soit $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ défini comme suit. Lequel parmi les énoncés suivants est* toujours *vrai, quel que soit le spécifique choix de f.*] **[5 x 2 pts]**

A. g may not be a function
B. g is a function, but it may neither be injective nor surjective
C. g is an injective function, but it might not be surjective
D. g is a surjective function, but it might not be injective
E. g is a bijective function

i) $g(x) = 1/f(x)$
ii) $g(x) = f^{-1}(x)$, where f^{-1} is the inverse of f
iii) $g(x) = \sqrt{f(x)}$
iv) $g(x) = |\sin(f(x))|$
v) $g(x) = \frac{1}{1+e^{f(x)}}$

Solution: A;E;E;B;C

4. Consider the following recursive computation of Fibonacci numbers. [*Considérez le suivant calcul récursif des nombres de Fibonacci.*] **[10pts]**

Function: `fib`(n)

Require: $n \in \mathbb{N}_{\geq 0}$
print "Again?"
if $n \leq 1$ **then**
    return 1
else
    return `fib`$(n-1)$ + `fib`$(n-2)$

The number of times the algorithm prints "Again?" is [*Le nombre de fois que l'algorithme affiche "Again?" est*]

A. $O(n^2 \log n)$ **B.** $O(n)$ **C.** $\Omega((3/2)^n)$ **D.** $\Omega(n!)$

Solution: C

5. Arrange the following functions in a list, so that each of them is big-O of the next function. **[12pts]** Write on the answer sheet the corresponding list of letters. [*Arrangez les fonctions suivantes en une liste, de telle manière que chaque fonction soit big-O de la prochaine. Écrivez sur la feuille de réponses la séquence de lettres correspondante*]

A. $\ln(n!)$ **B.** $(\lceil \ln n \rceil)!$ **C.** n **D.** $n(\ln n)^{2014}$ **E.** $5^{\sqrt{n}}$

Solution: C;A;D;B;E

PROBLEMS – write down the proofs on the answer sheet [*PROBLÈMES – écrivez les démonstrations sur la feuille de réponse*]

6. Prove that **[10pts]**

$$\sum_{i=0}^{n} \frac{1}{2i+1} \text{ is } \Theta(\log(n)).$$

Solution: On the one hand,

$$\sum_{i=0}^{n} \frac{1}{2i+1} \leq \sum_{i=1}^{2n+1} \frac{1}{i} \text{ which is } O(\log(2n+1)).$$

This means that there exist C and k so that $\sum_{i=1}^{2n+1} \frac{1}{i} \leq C\log(2n+1)$ for all $n \geq k \geq 1$. Hence, $\sum_{i=1}^{2n+1} \frac{1}{i} \leq C\log(2n+1) \leq C\log(3n) = C(\log(3)+\log(n)) \leq 2C\log(n)$ for all $n \geq \max\{k,3\}$. As a result, $\sum_{i=0}^{n} \frac{1}{2i+1}$ is $O(\log(n))$.
On the other hand,

$$\sum_{i=0}^{n} \frac{1}{2i+1} \geq \sum_{i=0}^{n} \frac{1}{2i+2} = \frac{1}{2}\sum_{i=0}^{n} \frac{1}{i+1} = \frac{1}{2}\sum_{i=1}^{n+1} \frac{1}{i} \geq \frac{1}{2}\sum_{i=1}^{n} \frac{1}{i}.$$

We have seen in class that $\sum_{i=1}^{n} \frac{1}{i}$ is $\Omega(\log(n)$ This means that there exist $C' > 0$ and k' so that $\sum_{i=0}^{n} \frac{1}{i} \geq C'\log(n)$ for $n \geq k'$. Hence, $\sum_{i=0}^{n} \frac{1}{2i+1} \geq \frac{C'}{2}\log(n)$ for $n \geq k'$, i.e., $\sum_{i=0}^{n} \frac{1}{2i+1}$ is $\Omega(\log(n))$.

7. Consider the Fibonacci sequence, i.e., the sequence defined by **[10pts]**

$$F_0 = 0,\ F_1 = 1,\ \text{and}\ F_n = F_{n-1} + F_{n-2},\ n \geq 2.$$

Show that $\gcd(F_{15}, F_{14}) = 1$.

Solution: Let us show more generally that $\gcd(F_n, F_{n-1}) = 1$, for any $n \geq 1$. For $n \geq 2$, $\gcd(F_n, F_{n-1}) = \gcd(F_{n-1}, F_n) = \gcd(F_{n-1}, F_{n-1} + F_{n-2}) = \gcd(F_{n-1}, F_{n-2})$. Using induction, this, together with $\gcd(F_1, F_0) = \gcd(1, 0) = 1$ as a base case, proves the claim.

8. Show that among every five points in a 1×1 square there exists at least one pair of points with distance less than or equal to $\frac{1}{\sqrt{2}}$. **[10 pts]**

Solution: Divide the square into four smaller $\frac{1}{2} \times \frac{1}{2}$ squares. By the pigeonhole principle, since there are four small squares but five points, there exists a small square containing at least two points. Further, any two points in a small square have distance at most $\frac{1}{\sqrt{2}}$. Therefore, there must exist at least one pair of points in the initial 1×1 square with distance less than or equal to $\frac{1}{\sqrt{2}}$.

9. Show that $\overline{A} \cap \overline{B} \cap \overline{C} = (\overline{B \cup C}) \setminus A$. **[10 pts]**

Solution: Starting from the left-hand side,

$$\begin{aligned}(\overline{B \cup C}) \setminus A &= (\overline{B} \cap \overline{C}) \setminus A \\ &= (\overline{B} \cap \overline{C}) \cap \overline{A} \\ &= \overline{A} \cap \overline{B} \cap \overline{C}.\end{aligned}$$

Here, in the first line we have used De Morgan's law $(\overline{E \cup F}) = \bar{E} \cap \bar{F}$. In the second step, we have used the fact that $E \setminus F = E \cap \bar{F}$. The final step follows since the operation of intersection is associative, so that we can drop the brackets.

10. Let $\mathcal{P}(A)$ denote the power set of the set A, i.e., the set of all subsets of A. Let $\mathcal{F}$ denote the set of functions from $\mathbb{N}$ to $\{0, 1\}$ (for example $f(n) := n \bmod 2$ is a member of $\mathcal{F}$). Show that $|\mathcal{P}(\mathbb{N})| = |\mathcal{F}|$. **[BONUS – 5 pts]**

Solution: We will show that there is an injective map from $\mathcal{P}(\mathbb{N})$ to $\mathcal{F}$ as well as an injective map from $\mathcal{F}$ to $\mathcal{P}(\mathbb{N})$. This will establish the desired bijection between the two sets and hence show that they have equal cardinality.
For any set $S \in \mathcal{P}(\mathbb{N})$ let the function $\chi_S : \mathbb{N} \to \{0, 1\}$ be defined as

$$\chi_S(n) := \begin{cases} 1, & \text{if } n \in S, \\ 0, & \text{otherwise.} \end{cases}$$

This is a function in $\mathcal{F}$ and for any two $S \neq S' \in \mathcal{P}(\mathbb{N})$ the corresponding function is different. This establishes the injection from $\mathcal{P}(\mathbb{N})$ to $\mathcal{F}$.
Conversely, let f be a function in $\mathcal{F}$ and let S_f be the subset of $\mathcal{P}(\mathbb{N})$ which contains the element n if and only if $f(n) = 1$. This map which maps f to S_f is again easily seen to be injective, establishing the claim.