Discrete Structures EPFL, Fall 2014

Partial Fractions

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We have seen that the generating function corresponding to a linear recurrence relation has the form $F(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and where $q(x)$ has a non-zero constant term. Further, we know that such a generating function corresponds to a formal power sum $F(x) = \sum_{n \geq 0} F_n x^n$. How can we find explicit expressions for F_n in an efficient manner? This is most easily done by using a *partical fraction decomposition.*

Before we discuss partial fractions, it is typically a good idea to compute a few terms F_0, F_1, F_2, "by hand" before proceeding further. It is easy to make a mistake in deriving the generating function and by checking the correctness of the first few terms we gain confidence that so far we did everything correctly and that the effort in computing the partial fractions is worth it.

How can we compute the first few terms efficiently? Assume at first that we want to compute the expansion corresponding to $\frac{1}{q(x)}$. Recall from our discussion that this is well defined if an only if $q(x)$ has a non-zero constant term. Further, in this case, computing this inverse is equivalent to the multiplicative inverse of $q(x)$. I.e., we are looking for a generating fuction $F(x) = \sum_{n \geq 0} F_n x^n$ so that

$$1 = q(x)F(x) = (\sum_{i \geq 0} q_i x^i)(\sum_{n \geq 0} F_n x^n).$$

A viable (although not very efficient way) of doing this is the following. Recall that if $H(x) = \sum_{n \geq 0} H_n x^n = q(x)F(x)$ then the coefficient H_n is given by $H_n = \sum_{i=0}^{n} q_i F_{n-i}$, the convolution of the coefficients. Since H_n by the above discussion is equal to 1 if $n = 0$ and 0 otherwise we get

$$\begin{aligned} q_0 F_0 = 1 &\leftrightarrow F_0 = \frac{1}{q_0}, \\ q_0 F_1 + q_1 F_0 = 0 &\leftrightarrow F_1 = -\frac{q_1 F_0}{q_0}, \\ &\cdots \\ \sum_{i=0}^{n} q_i F_{n-i} = 0 &\leftrightarrow F_n = -\frac{1}{q_0} \sum_{i=0}^{n-1} q_i F_{n-i}. \end{aligned}$$

If we have the slightly more general case where we want to compute the expansion of $p(x)/q(x)$ then this corresponds to finding a generating function $F(x)$ so that $p(x) = (\sum_{i \geq 0} q_i x^i)(\sum_{n \geq 0} F_n x^n)$. This is almost the same as before, but now we have a slightly more general $H(x)$, namely $H(x) = p(x)$. In this case we get

$$\begin{aligned} q_0 F_0 = p_0 &\leftrightarrow F_0 = \frac{p_0}{q_0}, \\ q_0 F_1 + q_1 F_0 = p_1 &\leftrightarrow F_1 = \frac{p_1 - q_1 F_0}{q_0}, \\ &\cdots \\ \sum_{i=0}^{n} q_i F_{n-i} = p_n &\leftrightarrow F_n = \frac{1}{q_0}(p_n - \sum_{i=0}^{n-1} q_i F_{n-i}). \end{aligned}$$

Rather than remembering these formulas, one can reformulate this calculation in a form that resembles the standard division algorithm. The only difference is that the "leading" terms in this division algorithm are the monomials of smallest degree rather than of highest degree. This is best explained by an explicit example.

Example 1. *Let* $F(x) = \frac{x^5}{8-20x+18x^2-7x^3+x^4}$. *We write down* x^5 *on the left and the denominator on the right. We now "divide" the left term by the right term where the "leading" terms are those of smallest degree. E.g., in the first step the leading term on the left is* x^5 *and the leading term on the right is* 8 *so that the quotient for this step is* $\frac{1}{8}x^5$. *We then compute the remainder and repeat this step with the remainder. In more detail,*

$$\begin{aligned} x^5 &= (8-20x+18x^2-7x^3+x^4).\frac{x^5}{8} + \frac{x^5}{8}(20x-18x^2+7x^3-x^4) \\ &= (8-20x+18x^2-7x^3+x^4).\frac{1}{8} + \underbrace{5/2x^6-9/4x^7+7/8x^8-1/8x^9}_{remainder}. \end{aligned}$$

The remainder is therefore $5/2x^6 - 9/4x^7 + 7/8x^8 - 1/8x^9$. *Note that the minimum degree of the remainder is* 6, *whereas the original degree of the numerator was* 5. *We now repeat this precedure on the remainder and at each step we pick the quotient in such a way that the lowest degree term on the left is cancelled. To see one more step, we get*

$$5/2x^6 - 9/4x^7 + 7/8x^8 - 1/8x^9 \quad = (8-20x+18x^2-7x^3+x^4).\frac{5}{16}x^6 - 9/4x^7 + 13/2x^8 - 37/16x^9 + 5/16x^{10}.$$

Note that again, the degree of the remainder is one larger than the degree we started with.

If we stop at this point we can conclude that

$$F(x) = \frac{x^5}{8-20x+18x^2-7x^3+x^4} = \frac{1}{8}x^5 + \frac{5}{16}x^6 - 9/4x^7 + 13/2x^8 - 37/16x^9 + 5/16x^{10},$$

which means that we have determined the first two non-zero terms of the expansion, namely $\frac{1}{8}x^5$ *and* $\frac{5}{16}x^6$.

So assume now that we have verified the expression for $F(x)$ and want to find explicit expressions for F_n. The method consists of two steps. First, we write $F(x)$ as a sum of "simple" expressions. This is the partial fraction expansions. Second, we recognize the expansions that correspond to each of these simple expansions and so we can directly write down the terms that correspond to each of these partial fractions.

Let us start with the partial fraction expansion. We first prove that such a representation always exists.

Theorem 1 (Existence of Partial Fraction Expansion)**.** *Let* $\frac{p(x)}{q(x)}$ *with deg*$(p) <$ *deg*(q) *and no common roots. Let* $q(x) = \prod_{i=1}^{n}(x-x_i)^{k_i}$, $k_i \in \mathbb{N}_{\geq 1}$, *and where all roots* x_i, $i = 1, \cdots, x_n$ *are distinct. Then, there exist unique constants* $A_{i,j}$ *so that*

$$\frac{p(x)}{q(x)} = \sum_{i=1}^{n}\sum_{j=0}^{k_i-1} \frac{A_{i,j}}{(x-x_i)^{k_i-j}}. \tag{1}$$

Proof. It suffices to show that for the given situation there exists a unique constant $A_{1,0}$ so that

$$\frac{p(x)}{q(x)} = \frac{A_{1,0}}{(x-x_1)^{k_1}} + \frac{p^*(x)}{(x-x_1)^{k_1-1}q^*(x)},$$

where $q^*(x) = \frac{q(x)}{(x-x_1)^k}$ and where $\deg(p^*) \leq \deg(q^*)+k_1-2$. The claim then follows by induction on the remaining term $\frac{p^*(x)}{(x-x_1)^{k_1-1}q^*(x)}$.

Multiply (1) by $q(x)$. We then get $p(x) = A_{1,0}q^*(x) + p^*(x)(x - x_1)$, or equivalently $p(x) - A_{1,0}q^*(x) = p^*(x)(x - x_1)$. Note that $p^*(x)$ is unspecified. So the only constraint that this equation implies is that $p(x) - A_{1,0}q^*(x)$ has a root at x_1. The latter is true if and only if we choose $A_{1,0} = p(x_1)/q^*(x_1)$. Note that by definition $q^*(x)$ does not have a root at x_1, so that this expression for $A_{1,0}$ is well defined. □

In principle we could find all the coefficients by following the steps of the above proof. But there is a much more efficient way.

Theorem 2 (Efficient Computation of Coefficients)**.** *Let $\frac{p(x)}{q(x)}$ with deg(p) $<$ deg(q) and no common roots. Let $q(x) = \prod_{i=1}^{n}(x - x_i)^{k_i}$, $k_i \in \mathbb{N}_{\geq 1}$, and where all roots x_i, $i = 1, \cdots, x_n$ are distinct so that*

$$\frac{p(x)}{q(x)} = \sum_{i=1}^{n}\sum_{j=0}^{k_i-1}\frac{A_{i,j}}{(x - x_i)^{k_i-j}}.$$

Then

$$A_{i,j} = \frac{1}{j!}\left(\frac{p(x)(x - x_i)^{k_i}}{q(x)}\right)^{(j)}\Bigg|_{x=x_i}.$$

Here, $(\cdot)^{(j)}$ denotes the j-th derivative. If $k_i = 1$, then $A_{i,0}$ can be computed even simpler as

$$A_{i,0} = \left.\frac{p(x)}{q'(x)}\right|_{x=x_i},$$

where $q'(x)$ denotes the derivative of $q(x)$.

Proof. Start again with (1) and multiply by $q(x)$. We get

$$p(x) = \sum_{i=1}^{n}\sum_{j=0}^{k_i-1}\frac{A_{i,j}q(x)}{(x - x_i)^{k_i-j}}. \tag{2}$$

Evaluate this equation at $x = x_i$. Note that $q(x)/(x - x_l)^{k_l-j}$ is a polynomial for every $1 \leq l \leq n$ and $0 \leq j \leq k_l - 1$ and that only if l is chosen to be equal to i does this polynomial not have a root at $x = x_i$. Hence, the evaluation gives us

$$p(x_i) = A_{i,0}\ \left(q(x)/(x - x_i)^{k_i}\right)\big|_{x=x_i}.$$

Solving for $A_{i,0}$ gives us the indicated formula. To get the expression for $A_{i,j}$, $j > 0$, look again at (2) and take the $(j - 1)$-th derivative before evaluating the expression again at $x = x_i$. This leads to the general expression. The simple expression for the case of a simple rule is a direct consequence of the fact that $\left(q(x)/(x - x_i)^{k_i}\right)\big|_{x=x_i} = q'(x_i)$ as can be checked by an explicit calculation. □

Note that in the previous two theorems we imposed the condition that deg(p) $<$ deg(q). This is not a serious restriction. If we have a rational function $F(x) = \frac{p(x)}{q(x)}$ with deg(p) $\geq$ deg(q) then simply find the unique polynomial $m(x)$ so that $p(x) - m(x)q(x) = r(x)$ has degree strictly smaller than the degree of $q(x)$. In other words, just cancel some high-degree terms from $p(x)$ by removing some suitable multiple of $q(x)$. This can always be done. Now write

$$F(x) = \frac{p(x)}{q(x)} = m(x) + \frac{r(x)}{q(x)},$$

and proceed with the term $\frac{r(x)}{q(x)}$ instead of $F(x)$ itself. The term $m(x)$ is already expanded as desired and hence can be left as is.

Let us now discuss the second step. We have seen that we can expand any rational $F(x)$ as sums of the form $\frac{A}{(x-x^*)^k}$, where A is a constant, x^* is the root and k is the multiplicity of the root,$j \in \mathbb{N}_{\geq 1}$. Such an expression can also be written as $\frac{A/(x^*)^k}{(1-x/x^*)^k}$. The $A/(x^*)^k$ is just a constant and instead of writing x/x^* we can absorb the factor $1/x^*$ into the x and just write x. Once the expansion is accomplished we can resurrect this factor by replacing every term x^n by $(1/x^*)^n x^n$. Therefore, the generic form that we need to expand is $\frac{1}{(1-x)^k}$.

Theorem 3 (Expansions Corresponding to Partial Fractions)**.** *Let $k \in \mathbb{N}_{\geq 1}$. Then*

$$\frac{1}{(1-x)^k} = \sum_{n\geq 0} \binom{n+k-1}{n} x^n$$

so that for any constant a

$$\frac{1}{(1-ax)^k} = \sum_{n\geq 0} \binom{n+k-1}{n} a^n x^n.$$

Proof. We know that $\frac{1}{1-x} = \sum_{n\geq 0} x^n$. If we take the $(k-1)$-th derivative on both sides we get $\frac{(k-1)!}{(1-x)^k} = \sum_{n\geq 0} n(n-1)\cdots(n-k+2)x^{n-k+1}$. Dividing both sides by $(k-1)!$ results in

$$\begin{aligned}
\frac{1}{(1-x)^k} &= \sum_{n\geq 0} \frac{n(n-1)\cdots(n-k+2)}{(k-1)!} x^{n-k+1} \\
&= \sum_{n\geq k-1} \frac{n!}{(k-1)!(n-k+1)!} x^{n-k+1} \\
&= \sum_{n\geq k-1} \binom{n}{n-k+1} x^{n-k+1} \\
&= \sum_{m\geq 0} \binom{m+k-1}{m} x^m \\
&= \sum_{n\geq 0} \binom{n+k-1}{n} x^n.
\end{aligned}$$

Hereby, the one-before-last step follows by setting $m = n - k + 1$. □

Example 2. *Let $F(x) = \frac{p(x)}{q(x)} = \frac{x^5}{8-20x+18x^2-7x^3+x^4} = \frac{x^5}{(x-1)(x-2)^3}$. We saw in our previous example that $F(x) = 1/8x^5 + 5/16x^6 + 1/2x^7 + \cdots$. Note that the denominator has degree 4 but the numerator has degree 5. We can therefore not directly apply the previous theorems. Therefore, let us first bring the numerator to the proper form by writing*

$$F(x) = \frac{x^5}{q(x)} = \frac{x^5 - (x+7)q(x)}{q(x)} = x + 7 + \frac{-56 + 132x - 106x^2 + 31x^3}{(x-1)(x-2)^3}.$$

We know from Theorem 1 that $F(x)$ has a unique representation of the form

$$7 + x + \frac{A_{1,0}}{(x-1)^1} + \frac{A_{2,0}}{(x-2)^3} + \frac{A_{2,1}}{(x-2)^2} + \frac{A_{2,2}}{(x-2)^1}.$$

Using the formulas in Theorem 2 we can now compute these coefficients. Since $x_1 = 1$ is a simple root we have

$$A_{1,0} = \left.\frac{-56 + 132x - 106x^2 + 31x^3}{q'(x)}\right|_{x=1} = \left.\frac{-56 + 132x - 106x^2 + 31x^3}{-20 + 36x - 21x^2 + 4x^3}\right|_{x=1} = -1.$$

For the root $x = 2$ we have to use the general formulas since this is a root of higher multiplicity. We have

$$A_{2,0} = \frac{1}{0!} \frac{-56 + 132x - 106x^2 + 31x^3}{x-1}\bigg|_{x=2} = 32,$$
$$A_{2,1} = \frac{1}{1!} \left(\frac{-56 + 132x - 106x^2 + 31x^3}{x-1}\right)'\bigg|_{x=2} = 48,$$
$$A_{2,2} = \frac{1}{2!} \left(\frac{-56 + 132x - 106x^2 + 31x^3}{x-1}\right)''\bigg|_{x=2} = 32.$$

Using the expansions from Theorem 3 we conclude that

$$\begin{aligned} F(x) &= 7 + x - \frac{1}{(x-1)^1} + \frac{32}{(x-2)^3} + \frac{48}{(x-2)^2} + \frac{32}{(x-2)^1} \\ &= 7 + x + \frac{1}{(1-x)^1} - \frac{4}{(1-x/2)^3} + \frac{12}{(1-x/2)^2} - \frac{16}{(1-x/2)^1} \\ &= 7 + x + \sum_{n\geq 0} x^n - 4\sum_{n\geq 0}\binom{n+2}{2}2^{-n}x^n + 12\sum_{n\geq 0}\binom{n+1}{1}2^{-n}x^n - 16\sum_{n\geq 0}2^{-n}x^n \\ &= 7 + x + \sum_{n\geq 0}(1 - 2^{-n+1}(4 + n(n-3)))x^n. \end{aligned}$$

If we compute the first few terms explicitly we see that this agrees with our previous expansion.