1. Vectors and the dot product

A vector $\vec{v}$ in $\mathbb{R}^3$ is an arrow. It has a direction and a length (aka the magnitude), but the position is not important. Given a coordinate axis, where the x-axis points out of the board, a little towards the left, the y-axis points to the right and the z-axis points upwards, there are three standard vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$, which have unit length and point in the direction of the x-axis, the y-axis and z-axis. Any vector in $\mathbb{R}^3$ may be written uniquely as a combination of these three vectors. For example, the vector $\vec{v} = 3\hat{\imath} - 2\hat{\jmath} + 4\hat{k}$ represents the vector obtained by moving 3 units along the x-axis, two units backwards along the y-axis and four units upwards.

If we imagine moving the vector so it's tail is at the origin then the endpoint P determines the vector. The point $P = (x, y, z)$ determines the vector $\vec{P} = \langle x, y, z\rangle$ starting at the origin and ending at the point P. Obviously,

$$\langle x, y, z\rangle = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \qquad \text{so that} \qquad \langle 3, -2, 4\rangle = 3\hat{\imath} - 2\hat{\jmath} + 4\hat{k}.$$

One advantage of this algebraic approach is that we can write down vectors in $\mathbb{R}^4$, for example, $\langle 2, 1, -3, 5\rangle$, $\langle \pi, \sin 2, -3, e^3\rangle$.

Question 1.1. *What is the direction of the zero vector which starts and ends at the origin?*

We will adopt the convention that the zero vector points in every direction. In coordinates the zero vector in $\mathbb{R}^3$ is given by $\langle 0, 0, 0\rangle$.

The length of the vector $\vec{v} = \langle a, b, c\rangle$ is the scalar

$$|\vec{v}| = (a^2 + b^2 + c^2)^{1/2}.$$

This is what you get if you apply Pythagoras' Theorem, twice.

One can add vectors in $\mathbb{R}^3$. If you want to add $\vec{u}$ and $\vec{v}$, move the starting point of $\vec{v}$ to the endpoint of $\vec{u}$; the sum is the arrow you get by first going along $\vec{u}$ and then along $\vec{v}$. To subtract two vectors is even easier. The vector $\vec{v} - \vec{u}$ is the vector starting at the endpoint of $\vec{u}$ and ending at the endpoint of $\vec{v}$.

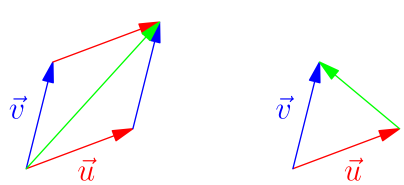

Figure 1. Addition and subtraction of vectors

Algebraically, it is easy to add vectors, add them component by component:

$$\begin{aligned}(3\hat{\imath} - 2\hat{\jmath} + 5\hat{k}) + (-4\hat{\imath} + 4\hat{\jmath} + 3\hat{k}) &= (3-4)\hat{\imath} + (-2+4)\hat{\jmath} + (5+3)\hat{k} \\ &= -\hat{\imath} + 2\hat{\jmath} + 8\hat{k}.\end{aligned}$$

More compactly,

$$\langle 3, -2, 5\rangle + \langle -4, 4, 3\rangle = \langle -1, 2, 8\rangle.$$

Note that it doesn't make sense to add a vector in $\mathbb{R}^2$ and a vector in $\mathbb{R}^3$. You can see this either algebraically or geometrically.

One can also multiply a scalar λ by a vector $\vec{v}$. $\lambda\vec{v}$ is the vector which is λ times as long as $\vec{v}$. If $\lambda > 0$, $\lambda\vec{v}$ has the same direction as $\vec{v}$ and if $\lambda < 0$, then $\lambda\vec{v}$ has the opposite direction. Either way, we will say that $\lambda\vec{v}$ is parallel to $\vec{v}$.

Algebraically, it is again easy to multiply a scalar by a vector,

$$\lambda\langle a, b, c\rangle = \langle \lambda a, \lambda b, \lambda c\rangle \qquad \text{so that} \qquad -3\langle 1, 2, -3\rangle = \langle -3, -6, 9\rangle.$$

The direction is what is left after you remove the length,

$$\hat{u} = \frac{\vec{v}}{|\vec{v}|}.$$

Note that $\hat{u}$ is a unit vector; it's length is one. Vectors will always have arrows on top of them, unit vectors hats.

We can always write a vector as a product of its length times its direction,

$$\vec{v} = |\vec{v}|\left(\frac{\vec{v}}{|\vec{v}|}\right).$$

For example,

$$\langle 1, -2, 2\rangle = 3\langle 1/3, -2/3, 2/3\rangle \qquad \text{and} \qquad \langle 3, 4\rangle = 5\langle 3/5, 4/5\rangle.$$

Question 1.2. *Let M be the midpoint of the line segment AB. Find the vector $\vec{M}$ in terms of the vectors $\vec{A}$ and $\vec{B}$.*

To get to M, from A, one has to go half way from A to B. The vector from A to B is $\overrightarrow{AB} = \vec{B} - \vec{A}$. Halfway means

$$\frac{1}{2}(\vec{B} - \vec{A}),$$

and so this is the vector from A to M. Therefore

$$\vec{M} = \vec{A} + \overrightarrow{AM} = \overrightarrow{A} + \frac{1}{2}(\vec{B} - \vec{A}) = \frac{1}{2}(\vec{A} + \vec{B}).$$

Question 1.3. *Show that the diagonals of a parallelogram bisect each other.*

Let's give names to the usual suspects. Let's call the vertices of the parallelogram A, B, C and D. Let X and Y be the midpoints of the diagonals. It is enough to show that $X = Y$ (naming the midpoints of the diagonals is the sneakiest part of the solution to this problem). What do we know? Well, since we have a parallelogram,

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{CD} \\ \vec{B} - \vec{A} &= \vec{D} - \vec{C} \\ \vec{B} + \vec{C} &= \vec{A} + \vec{D}.\end{aligned}$$

(assuming we have labelled the vertices appropriately). We have

$$\vec{X} = \frac{1}{2}(\vec{A} + \vec{D}) \qquad \text{and} \qquad \vec{Y} = \frac{1}{2}(\vec{B} + \vec{C}).$$

So

$$\begin{aligned}\vec{Y} &= \frac{1}{2}(\vec{B} + \vec{C}) \\ &= \frac{1}{2}(\vec{A} + \vec{D}) \\ &= \vec{X}.\end{aligned}$$

Since the vectors $\vec{X}$ and $\vec{Y}$ both start at the origin we must have $X = Y$, which is what we want.

Question 1.4. *How do we multiply two vectors?*

Actually there are two answers to this question. The first answer is to take the dot product. If the vectors are

$$\vec{v}_1 = \langle a_1, b_1, c_1 \rangle \qquad \text{and} \qquad \vec{v}_2 = \langle a_2, b_2, c_2 \rangle,$$

then the dot product is the scalar

$$\vec{v}_1 \cdot \vec{v}_2 = a_1 a_2 + b_1 b_2 + c_1 c_2.$$

For example,

$$\langle 1, -2, 4 \rangle \cdot \langle 3, 1, -2 \rangle = 1 \cdot 3 + 1 \cdot -2 + 4 \cdot -2 = 3 - 2 - 8 = -7.$$

Note that

$$\vec{v} \cdot \vec{v} = |v|^2.$$

The usual rules of algebra apply to the dot product:

(1) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.
(2) $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$.
(3) $(\lambda \vec{u}) \cdot \vec{v} = \lambda(\vec{u} \cdot \vec{v})$.

Theorem 1.5 (Geometric interpretation of the dot product)**.** *If θ is the angle between the two vectors $\vec{u}$ and $\vec{v}$, then*

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \theta.$$

Proof. If either $\vec{u}$ or $\vec{v}$ is the zero vector, then both sides are zero, and we certainly have equality (and we can take θ to be any angle we please, which is consistent with our convention that the zero vector points in every direction). So we may assume that $\vec{u}$ and $\vec{v}$ are both non-zero. If $\vec{u}$ and $\vec{v}$ are parallel, then $\theta = 0$ or π and it is straightforward to check that both sides are equal.

Otherwise, let $\vec{w} = \vec{v} - \vec{u}$, the third side of the triangle with sides given by $\vec{u}$ and $\vec{v}$. Then the square of the length of the third side is

$$\begin{aligned} |\vec{w}|^2 &= \vec{w} \cdot \vec{w} \\ &= (\vec{v} - \vec{u}) \cdot (\vec{v} - \vec{u}) \\ &= |\vec{v}|^2 + |\vec{u}|^2 - 2\vec{u} \cdot \vec{v}. \end{aligned}$$

Compare this with the formula given by the cosine rule. If the lengths of the three sides are are u, v and w, the cosine rule says,

$$w^2 = u^2 + v^2 - 2uv \cos \theta.$$

Now $u = |\vec{u}|$, $v = |\vec{v}|$ and $w = |\vec{w}|$, so putting these two formulae side by side, we see

$$\begin{aligned} w^2 &= u^2 + v^2 - 2\vec{u} \cdot \vec{v} \\ w^2 &= u^2 + v^2 - 2uv \cos \theta, \end{aligned}$$

so that subtracting we get

$$0 = 2(\vec{u} \cdot \vec{v} - uv \cos \theta),$$

whence the result. □

The virtue of (1.5) is that we can use it to find the angle between two vectors.

Question 1.6. *Consider the triangle in space with vertices $A = (1, 0, 0)$, $B = (1, 1, -1)$ and $C = (-1, 1, 0)$.*

What is the angle at A?

Let $\vec{u} = \overrightarrow{AB} = \langle 0, 1, -1 \rangle$ and $\vec{v} = \overrightarrow{AC} = \langle -2, 1, 0 \rangle$. We want the angle θ between $\vec{u}$ and $\vec{v}$. Well,

$$\begin{aligned}\cos\theta &= \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} \\ &= \frac{\langle 0,1,-1\rangle \cdot \langle -2,1,0\rangle}{|\langle 0,1,-1\rangle||\langle -2,1,0\rangle|} \\ &= \frac{1}{\sqrt{2}\cdot\sqrt{5}} \\ &= \frac{1}{\sqrt{10}}.\end{aligned}$$

In this case

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{10}}\right) \approx 1.25$$

radians, which in degrees is about 71.57.

What can we say about the sign of

$$\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}?$$

If this is positive we have an angle less than $\pi/2$ and if this is negative an angle greater than $\pi/2$. It is zero if and only if the angle is $\pi/2$ and the vectors are orthogonal.

Question 1.7. *Fix a point Q and a vector $\vec{n}$. What is the set of points $P = (x, y, z)$ such that the vector $\overrightarrow{PQ}$ is orthogonal to a vector $\vec{n}$?*

This is a plane.

Question 1.8. *What is the set of points where $2x - y + 3z = 0$?*

One way to answer this question is to guess using an analogy. If we were to drop a variable, we'd get $2x - y = 0$, which represents a line through the origin in $\mathbb{R}^2$. A reasonable guess is that this represents a plane through the origin.

Suppose we put $\vec{n} = \langle 2, -1, 3 \rangle$. Let $P = (x, y, z)$ so that $\vec{P} = \langle x, y, z \rangle$. Then

$$\vec{P}\cdot\vec{n} = \langle x, y, z\rangle \cdot \langle 2, -1, 3\rangle = 2x - y + 3z.$$

The condition that this is zero, represents the condition that the vector $\vec{P}$ is orthogonal to $\vec{n}$. So this represents the plane through the origin orthogonal to the vector $\langle 2, -1, 3 \rangle$.

Let $\vec{F}$ represent a force. Notice that this makes sense; forces have a direction and a magnitude.

Question 1.9. *What is the component of the force $\vec{F}$ in the direction $\hat{u}$ (this is a direction, so $\hat{u}$ is a unit vector)?*

This is a scalar, a number. If one draws a triangle, with hypotenuse given by $\vec{F}$ and one side parallel to $\hat{u}$ and θ is the angle between $\vec{F}$ and $\hat{u}$ we want the length of the adjacent side. By the usual rules for trigonometry this is the length of the hypotenuse times the cosine of the angle θ, that is $|\vec{F}|\cos\theta$. But $\hat{u}$ has length one, so that $|\hat{u}| = 1$. So the component of $\vec{F}$ in the direction $\hat{u}$ is the dot product $\vec{F} \cdot \hat{u}$.

Even if $\vec{F}$ is not a force, one can always take the component of $\vec{F}$ in the direction of $\vec{u}$.