14. Non-independent variables

Consider an ideal gas, which has a pressure P, a volume V and a temperature T. These variables are not independent, they are related by

$$PV = kT \qquad \text{where} \qquad k = nR.$$

Here n and R are constants, which depend on the amount and type of gas.

Many functions depend on all three variables, and finding out how the function varies as one changes the variables is complicated. Let's consider an easy example.

Question 14.1. *Suppose we have a right-angled triangle with sides x, y and hypotenuse z. Then*

$$x^2 + y^2 = z^2,$$

so x, y and z are not independent. Let $P = x + y + z$ the length of the perimeter. What is

$$\frac{\partial P}{\partial x}?$$

There are (at least) four possible answers.

(1) 1, since y and z are constant.
(2) Rewrite as
$$x + y + \sqrt{x^2 + y^2},$$
and take the partial with respect to x (fixing y).
(3) Rewrite as
$$x + \sqrt{z^2 - x^2} + z,$$
and take the partial with respect to x (fixing z).
(4) The question is ambiguous.

The answer cannot be one, since if we vary x, one of at least y or z has to vary. We have to choose to hold either y or z constant.

If we fix y and increase x, then we get wider triangles of the same height. If we fix z and increase x, then our triangle is like a ladder sliding down the wall. So the answer is (4), the question is ambiguous.

To make the question unambiguous, we have to specify what variables we are holding constant. New notation:

$$\left(\frac{\partial P}{\partial x}\right)_y.$$

This means, vary x whilst holding y constant. Implicitly this means we treat z as a function of x and y.

There are three methods to compute this derivative. Perhaps the best is just to use to the total differential:

$$\mathrm{d}P = P_x\,\mathrm{d}x + P_y\,\mathrm{d}y + P_z\,\mathrm{d}z.$$

For us, this means

$$\mathrm{d}P = \mathrm{d}x + \mathrm{d}y + \mathrm{d}z.$$

Now let's differentiate the relation:

$$2x\,\mathrm{d}x + 2y\,\mathrm{d}y = 2z\,\mathrm{d}z.$$

Solving for $\mathrm{d}z$,

$$\mathrm{d}z = \frac{x}{z}\,\mathrm{d}x + \frac{y}{z}\,\mathrm{d}y.$$

Substitute this back into the expression for dP,

$$\begin{aligned}\mathrm{d}P &= \mathrm{d}x + \mathrm{d}y + \mathrm{d}z \\ &= \mathrm{d}x + \mathrm{d}y + \left(\frac{x}{z}\,\mathrm{d}x + \frac{y}{z}\,\mathrm{d}y\right) \\ &= (1 + \frac{x}{z})\,\mathrm{d}x + (1 + \frac{y}{z})\,\mathrm{d}y.\end{aligned}$$

So

$$\left(\frac{\partial P}{\partial x}\right)_y = 1 + \frac{x}{z}.$$

Note that we also computed

$$\left(\frac{\partial P}{\partial y}\right)_x = 1 + \frac{y}{z},$$

at the same time.

There are two other methods to compute the derivative. We could eliminate z

$$x + y + \sqrt{x^2 + y^2},$$

and differentiate with respect to x, fixing y,

$$1 + \frac{x}{\sqrt{x^2 + y^2}}.$$

Finally, we could use the chain rule.

$$P = x + y + z.$$

Differentiate both sides with respect to x, fixing y,

$$\left(\frac{\partial P}{\partial x}\right)_y = 1 + \left(\frac{\partial z}{\partial x}\right)_y.$$

So we need to calculate the last term. We know

$$x^2 + y^2 = z^2.$$

Differentiate both sides with respect to x, fixing y,

$$2x = 2z \left(\frac{\partial z}{\partial x}\right)_y .$$

Solving, we get

$$\left(\frac{\partial z}{\partial x}\right)_y = \frac{x}{z}.$$

Putting all of this together, we get

$$\left(\frac{\partial P}{\partial x}\right)_y = 1 + \frac{x}{z}.$$

Example 14.2. *Suppose we have a triangle, sides a, b and angle θ between the sides. The area of the triangle is*

$$A = \frac{1}{2} ab \sin\theta.$$

Suppose also that the triangle is a right-angled triangle, with b the hypotenuse, so that

$$a = b\cos\theta.$$

What is

$$\frac{\partial A}{\partial \theta}?$$

Once again this question is ambiguous. If we forget that the triangle is right-angled, we get

$$\frac{\partial A}{\partial \theta} = \frac{1}{2} ab\cos\theta.$$

But now suppose we vary θ keeping a right-angled triangle. There are two quantities we could compute

$$\left(\frac{\partial A}{\partial \theta}\right)_a .$$

We keep a right-angled triangle and fix one side. Or we could compute

$$\left(\frac{\partial A}{\partial \theta}\right)_b .$$

We keep a right-angled triangle and fix the hypotenuse.

Just to practice, let's compute the first expression. We first use the method of differentials.

$$\begin{aligned} \mathrm{d}A &= A_a \,\mathrm{d}a + A_b \,\mathrm{d}b + A_\theta \,\mathrm{d}\theta \\ &= \frac{1}{2} b\sin\theta \,\mathrm{d}a + \frac{1}{2} a\sin\theta \,\mathrm{d}b + \frac{1}{2} ab\cos\theta \,\mathrm{d}\theta. \end{aligned}$$

If we differentiate the relation, we get

$$\mathrm{d}a = \cos\theta\,\mathrm{d}b - b\sin\theta\,\mathrm{d}\theta.$$

Now we want to view A as a function of θ and a. So we want to get rid of $\mathrm{d}b$,

$$\mathrm{d}b = \sec\theta\,\mathrm{d}a + b\tan\theta\,\mathrm{d}\theta.$$

So

$$\begin{aligned}
\mathrm{d}A &= \frac{1}{2}b\sin\theta\,\mathrm{d}a + \frac{1}{2}a\sin\theta\,\mathrm{d}b + \frac{1}{2}ab\cos\theta\,\mathrm{d}\theta. \\
&= \frac{1}{2}b\sin\theta\,\mathrm{d}a + \frac{1}{2}a\sin\theta\,(\sec\theta\,\mathrm{d}a + b\tan\theta\,\mathrm{d}\theta) + \frac{1}{2}ab\cos\theta\,\mathrm{d}\theta. \\
&= \frac{1}{2}(a\tan\theta + b\sin\theta)\,\mathrm{d}a + \frac{1}{2}ab(\cos\theta + \sin\theta\tan\theta)\,\mathrm{d}\theta.
\end{aligned}$$

Hence

$$\left(\frac{\partial A}{\partial\theta}\right)_a = \frac{1}{2}ab(\cos\theta + \sin\theta\tan\theta).$$

Next, let's try substitution.

$$b = a\sec\theta.$$

So,

$$A = \frac{1}{2}a^2\tan\theta.$$

Hence

$$\left(\frac{\partial A}{\partial\theta}\right)_a = \frac{1}{2}a^2\sec^2\theta.$$

Let's check the two answers are compatible.

$$\frac{1}{2}ab(\cos\theta + \sin\theta\tan\theta) = \frac{1}{2}a^2(1 + \tan^2\theta) = \frac{1}{2}a^2\sec^2\theta.$$

Finally, let's use the chain rule.

$$\left(\frac{\partial A}{\partial\theta}\right)_a = \frac{1}{2}a\left(\frac{\partial b}{\partial\theta}\right)_a\sin\theta + \frac{1}{2}ab\cos\theta.$$

Here we applied the product rule to the product $b\sin\theta$, treating $a/2$ as a constant. So we need to compute the derivative on the RHS. We differentiate the relation,

$$0 = \left(\frac{\partial b}{\partial\theta}\right)_a\cos\theta - b\sin\theta,$$

where again we applied the product rule. It follows that

$$\left(\frac{\partial b}{\partial\theta}\right)_a = b\tan\theta.$$

Putting all of this together, we get

$$\left(\frac{\partial A}{\partial \theta}\right)_a = \frac{1}{2}ab\tan\theta\sin\theta + \frac{1}{2}ab\cos\theta.$$