17. Review

Linear approximation:

$$\Delta f \approx f_x \Delta x + f_y \Delta y.$$

Tangent plane: to $z = f(x, y)$ at (x_0, y_0, z_0)

$$z - z_0 = f_x(x - x_0) + f_y(y - y_0).$$

Let $w = f(x, y, z)$. Chain rule:

$$\mathrm{d}w = f_x \,\mathrm{d}x + f_y \,\mathrm{d}y + f_z \,\mathrm{d}z.$$

So

$$\frac{dw}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt}$$

We can encode this efficiently using the gradient:

$$\nabla f = \langle f_x, f_y, f_z \rangle = f_x \hat{\imath} + f_y \hat{\jmath} + f_z \hat{k},$$

Then

$$\frac{dw}{dt} = \nabla f \cdot \vec{v}(t).$$

The most important property of the gradient is that it is normal to the level curves, or to the level surfaces.

Example 17.1. *What is the tangent plane to the ellipsoid*

$$3x^2 + 5y^2 + 3z^2 = 11,$$

at the point $(x_0, y_0, z_0) = (1, 1, 1)$*?*

Well, this is a level surface of the function $f(x, y, z) = 3x^2 + 5y^2 + 3z^2$.

$$\nabla f = \langle 6x, 10y, 6z \rangle.$$

At the point $(1, 1, 1)$, we have

$$\nabla f = \langle 6, 10, 6 \rangle.$$

So $\vec{n} = \langle 3, 5, 3 \rangle$ is a normal vector the tangent plane. So the equation of the tangent plane is

$$\langle x-1, y-1, z-1 \rangle \cdot \langle 3, 5, 3 \rangle = 0 \qquad \text{so that} \qquad 3(x-1) + 5(y-1) + 3(z-1) = 0.$$

Rearranging, we get $3x + 5y + 3z = 11$.

Directional derivative: Let $w = f(x, y)$ be a function of two variables. Let $\hat{u} = \langle a, b \rangle$ be a direction in the plane. The directional derivative, in the direction of $\hat{u}$,

$$\left.\frac{dw}{ds}\right|_{\hat{u}} = \lim_{s \to 0} \frac{f(x_0 + sa, y_0 + sb) - f(x_0, y_0)}{s}.$$

If $\hat{u} = \hat{\imath}$, we get $f_x(x_0, y_0)$ and if $\hat{u} = \hat{\jmath}$, we get $f_y(x_0, y_0)$.

To compute, use the gradient:

$$\left.\frac{dw}{ds}\right|_{\hat{u}} = \nabla f \cdot \hat{u}.$$

So the gradient points in the direction of maximum increase of w and the magnitude of the gradient is the rate of change in this direction. The direction of maximum decrease of f is given by $-\nabla f$.

Example 17.2. *What is the closest point to $p = (1, -1)$ on the curve $x^3 - x + 2y^2 = 1.9$?*

At $(1, -1)$ we have $f(1, -1) = 2$, so we want $\Delta f = -0.1$. From p we should go in the direction to decrease f the most:

$$\nabla f = \langle 3x^2 - 1, 4y\rangle \qquad \text{so that} \qquad \nabla f_{(1,-1)} = \langle 2, -4\rangle.$$

We want go in the direction of $-\nabla f_{(1,-1)} = \langle -2, 4\rangle$. The magnitude is $2\sqrt{5}$, so want to go in the direction

$$\hat{u} = \frac{1}{\sqrt{5}}\langle -1, 2\rangle.$$

If we go in this direction f decreases by $2\sqrt{5}$. So we want to go a distance of

$$\frac{1}{20\sqrt{5}}.$$

That is we want a displacement of

$$\frac{1}{100}\langle -1, 2\rangle.$$

So we want the point

$$\langle 1, -1\rangle + \frac{1}{100}\langle -1, 2\rangle = \langle 0.99, -0.98\rangle$$

To find the maximum and the minimum of a function $w = f(x, y)$, first find the critical points, the solutions to $f_x = 0$ and $f_y = 0$. To analyse the type of the critical points (local minimum, local maximum or saddle point), use the 2nd derivative test. Let $A = f_{xx}(x_0, y_0)$, $B = f_{xy}(x_0, y_0)$ and $C = f_{yy}(x_0, y_0)$. If $AC - B^2 > 0$ we have a maximum or minimum. $A > 0$ is a minimum and $A < 0$ is a maximum. If $AC - B^2 < 0$ we have a saddle point.

Next check what happens at the boundary, including infinity.

Example 17.3.

maximise and minimise $\quad x+y+z \quad$ *subject to* $\quad x^2y^3z^5 = 2^2 3^3 5^5$,

where x, y and $z \geq 0$.

Use equation to eliminate x,

$$x = \sqrt{\frac{2^2 3^3 5^5}{y^3 z^5}}.$$

So we want to maximise

$$h(y,z) = \sqrt{\frac{2^2 3^3 5^5}{y^3 z^5}} + y + z.$$

Find the critical points:

$$h_y = -\frac{3}{2}\sqrt{\frac{2^2 3^3 5^5}{y^5 z^5}} + 1 \qquad \text{and} \qquad h_z = -\frac{5}{2}\sqrt{\frac{2^2 3^3 5^5}{y^3 z^7}} + 1.$$

So we want

$$0 = -\frac{3}{2}\sqrt{\frac{2^2 3^3 5^5}{y^5 z^5}} + 1 \qquad \text{and} \qquad 0 = -\frac{5}{2}\sqrt{\frac{2^2 3^3 5^5}{y^3 z^7}} + 1.$$

Rearranging, we get

$$\sqrt{\frac{2^2 3^3 5^5}{y^5 z^5}} = \frac{2}{3} \qquad \text{and} \qquad \sqrt{\frac{2^2 3^3 5^5}{y^3 z^7}} = \frac{2}{5}.$$

Squaring, we get

$$\frac{2^2 3^3 5^5}{y^5 z^5} = \frac{2^2}{3^2} \qquad \text{and} \qquad \frac{2^2 3^3 5^5}{y^5 z^7} = \frac{2^2}{5^2}.$$

Taking the reciprocal

$$\frac{y^5 z^5}{2^2 3^3 5^5} = \frac{3^2}{2^2} \qquad \text{and} \qquad \frac{y^5 z^7}{2^2 3^3 5^5} = \frac{5^2}{2^2}.$$

Simplifying

$$y^5 z^5 = 3^5 5^5 \qquad \text{and} \qquad y^3 z^7 = 3^3 5^7.$$

We guess $y = 3$ and $z = 5$. This works and it is clear the solution is unique. $x = 2$ is the other value. Let's try the 2nd derivative test.

$$h_{yy} = \frac{15}{4}\sqrt{\frac{2^2 3^3 5^5}{y^7 z^5}} \qquad h_{yz} = \frac{15}{4}\sqrt{\frac{2^2 3^3 5^5}{y^5 z^7}} \qquad \text{and} \qquad h_{zz} = \frac{35}{4}\sqrt{\frac{2^2 3^3 5^5}{y^3 z^9}}.$$

We have

$$A = \frac{15}{4}\sqrt{\frac{2^2 3^3 5^5}{3^7 5^5}} \qquad B = \frac{15}{4}\sqrt{\frac{2^2 3^3 5^5}{3^5 5^7}} \qquad \text{and} \qquad C = \frac{35}{4}\sqrt{\frac{2^2 3^3 5^5}{3^3 5^9}}.$$

We have $AC - B^2 > 0$. $A > 0$, so have a local minimum. There are no other critical points, so this is a global minimum. Minimum value is 10.

At the boundary, one of the variables goes to ∞ and the sum goes to ∞. No maximum.

Let's use Lagrange multipliers insead. We add a variable λ and solve

$$\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ f_z &= \lambda g_z \\ g &= c. \end{aligned}$$

In our case $f(x, y, z) = x + y + z$ and $g(x, y, z) = x^2 y^3 z^5$. We get

$$\begin{aligned} 1 &= \lambda 2 x y^3 z^5 \\ 1 &= \lambda 3 x^2 y^2 z^5 \\ 1 &= \lambda 5 x^2 y^3 z^4 \\ x^2 y^3 z^5 &= 2^2 3^3 5^5. \end{aligned}$$

Multiply the first three equations by x, y and z:

$$\begin{aligned} x &= \lambda 2 x^2 y^3 z^5 \\ y &= \lambda 3 x^2 y^3 z^5 \\ z &= \lambda 5 x^2 y^3 z^5. \end{aligned}$$

So $3x = 2y$, $5x = 2z$. Multiply constraint by 2^3,

$$3^3 x^5 z^5 = 2^5 3^3 5^5.$$

Cancelling, we get

$$x^5 z^5 = 2^5 5^5.$$

Multiply both sides by 2^5,

$$2^5 x^5 z^5 = 2^{10} 5^5.$$

We get

$$5^5 x^{10} = 2^{10} 5^5.$$

Hence $x = 2$. Thus $y = 3$ and $z = 5$.