19. Vector fields

Suppose we have a function

$$\vec{F} = M\hat{\imath} + N\hat{\jmath},$$

which assigns to every point of the plane a vector. Here $M = M(x, y)$ and $N(x, y)$ are scalar functions, which are the components of $\vec{F}$. Such a function $\vec{F}$ is called a vector field.

Vector fields appear in many different guises. If you look at the flow of water in a river, every point of the river has a velocity vector. When the wind blows, it blows in different directions and different speeds at every point. Force often forms a vector field. Gravitation always pulls you to the centre of the earth, inversely proportional to the square of the distance to the origin.

It is interesting to draw pictures of vector fields;

(1) $\vec{F} = \hat{\imath} - 3\hat{\jmath}$.
(2) $\vec{F} = x\hat{\imath}$.
(3) $\vec{F} = x\hat{\imath} + y\hat{\jmath}$.
(4) $\vec{F} = -y\hat{\imath} + x\hat{\jmath}$.

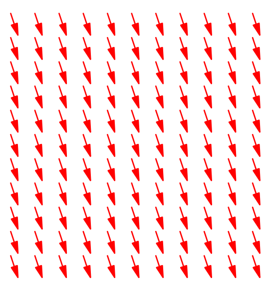

Figure 1. Picture of $\vec{F} = \hat{\imath} - 3\hat{\jmath}$

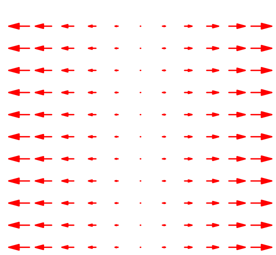

Figure 2. Picture of $\vec{F} = x\hat{\imath}$

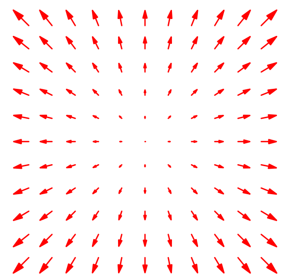

FIGURE 3. Picture of $\vec{F} = x\hat{\imath} + y\hat{\jmath}$

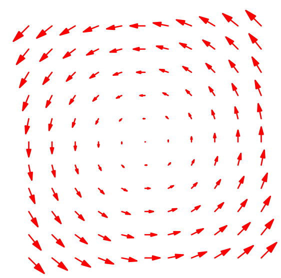

FIGURE 4. Picture of $\vec{F} = -y\hat{\imath} + x\hat{\jmath}$

Recall that the work done is the dot product of the force and the displacement,

$$W \approx \vec{F} \cdot \Delta\vec{r},$$

for a small displacement $\Delta\vec{r}$. If we sum all of the displacements along a trajectory C, we get a Riemann sum and taking the limit as $\Delta\vec{r}$ goes to zero, we get an integral, called a line integral

$$W = \int_C \vec{F} \cdot \mathrm{d}\vec{r} = \lim_{\Delta\vec{r}\to 0} \sum_i \vec{F}_i \cdot \Delta\vec{r}_i.$$

To calculate the line integral, choose a parametrisation $\vec{r}(t)$ of C (you can think of t as the time);

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_{t_1}^{t_2} \left(\vec{F} \cdot \frac{d\vec{r}}{dt}\right) \mathrm{d}t.$$

For example, suppose $\vec{F} = -y\hat{\imath} + x\hat{\jmath}$ and C is given by $x = t$ and $y = t^2$, $0 \le t \le 1$. So C is part of the parabola $y = x^2$, starting at $(0,0)$ and ending at $(1,1)$.

We calculate everything in terms of t,

$$\vec{F} = \langle -y, x\rangle = \langle -t^2, t\rangle \qquad \text{and} \qquad \frac{d\vec{r}}{dt} = \langle 1, 2t\rangle.$$

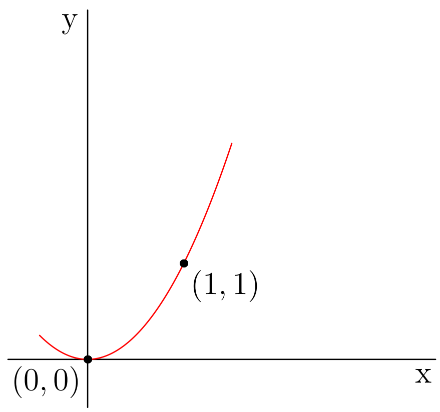

FIGURE 5. The curve C

Hence

$$\int_C \vec{F}\cdot \mathrm{d}\vec{r} = \int_0^1 \vec{F}\cdot \frac{d\vec{r}}{dt}\,\mathrm{d}t = \int_0^1 \langle -t^2, t\rangle \cdot \langle 1, 2t\rangle\,\mathrm{d}t = \int_0^1 t^2\,\mathrm{d}t = \left[\frac{t^3}{3}\right]_0^1 = \frac{1}{3}.$$

Note that we could parametrise C in many different ways. For example, we could choose $x = \sin\theta$, $y = \sin^2\theta$. In this case,

$$\vec{F} = \langle -y, x\rangle = \langle -\sin^2\theta, \sin\theta\rangle \qquad \text{and} \qquad \frac{d\vec{r}}{dt} = \langle \cos\theta, 2\cos\theta\sin\theta\rangle.$$

Hence

$$\int_C \vec{F}\cdot\mathrm{d}\vec{r} = \int_0^{\pi/2} \langle -\sin^2\theta, \sin\theta\rangle\cdot\langle \cos\theta, 2\cos\theta\sin\theta\rangle\,\mathrm{d}\theta = \int_0^{\pi/2} \cos\theta\sin^2\theta\,\mathrm{d}\theta.$$

If we make the substitution $t = \sin\theta$ we get back to the old integral. In practice we always try to use the simplest parametrisation.

There is an alternative and quite pervasive notation for line integrals. We have $\vec{F} = \langle M, N\rangle$ and $\mathrm{d}\vec{r} = \langle \mathrm{d}x, \mathrm{d}y\rangle$. So the line integral is

$$\int_C \vec{F}\cdot \mathrm{d}\vec{r} = \int_C M\mathrm{d}x + N\mathrm{d}y.$$

Note that this notation is a little confusing; it is important to realise we still have a line integral. In the example above we have

$$\int_C -y\,\mathrm{d}x + x\,\mathrm{d}y = \int_0^1 -t^2\,\mathrm{d}t + t\,\mathrm{d}t^2 = \int_0^1 t^2\,\mathrm{d}t^2 = \frac{1}{3}.$$

Here we used the fact that

$$\mathrm{d}y = \mathrm{d}t^2 = 2t\,\mathrm{d}t.$$

Sometimes it is better to use the arclength parametrisation. Recall we have

$$\frac{d\vec{r}}{dt} = \frac{ds}{dt}\hat{T},$$

where s is the arclength parameter and $\hat{T}$ is the unit tangent vector. So

$$\mathrm{d}\vec{r} = \hat{T}\mathrm{d}s.$$

In this case,

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_C \vec{F} \cdot \hat{T}\,\mathrm{d}s.$$

For example, suppose C is a circle of radius a centred at the origin. If $\vec{F} = x\hat{\imath} + y\hat{\jmath}$, then $\vec{F}$ is orthogonal to the unit tangent vector, so that

$$\vec{F} \cdot \hat{T} = 0.$$

In this case,

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_C \vec{F} \cdot \hat{T}\,\mathrm{d}s = 0.$$

Another way to think of this is as follows. If the force is radial then it is perpendicular to the direction in which we move, so the work done is zero.

Now suppose that $\vec{F} = -y\hat{\imath} + x\hat{\jmath}$. Then $\vec{F} \cdot \hat{T} = a$, the magnitude of $\vec{F}$. In this case

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_C a\,\mathrm{d}s = 2\pi a^2.$$

On the other hand, let's choose the parametrisation $x = a\cos\theta$, $y = a\sin\theta$. Then

$$\vec{F} = \langle -a\sin\theta, a\cos\theta\rangle \qquad \text{and} \qquad \frac{d\vec{r}}{dt} = \langle -a\sin\theta, a\cos\theta\rangle.$$

So we get

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_0^{2\pi} a^2\mathrm{d}\theta = 2\pi a^2.$$