2. Cross products

Let's suppose you want to calculate the area of a polygon in the plane. Nothing easier, break the polygon into triangles and calculate the area of each triangle.

Question 2.1. *What is the area of a triangle, with two sides determined by the vectors $\vec{u}$ and $\vec{v}$?*

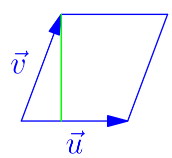

Figure 1. Area of parallelogram

The area of a triangle is 1/2 base $\times$ height, which is half of the area of the corresponding parallelogram, where $|\vec{u}|$ is the base and the length of the green line is the height, $|\vec{v}|\sin\theta$. The area of the parallelogram is $|\vec{u}||\vec{v}|\sin\theta$. How to get our hands on $\sin\theta$? One could use the identity,

$$\cos^2\theta + \sin^2\theta = 1,$$

to get a formula for $\sin\theta$ in terms of $\cos\theta$ and use the dot product, but it is pretty clear that it is going to give an ugly formula.

We prefer cosines to sines, since cosines turn up in dot products. If we have the complementary angle

$$\phi = \frac{\pi}{2} - \theta,$$

then we could use the fact that

$$\sin\theta = \cos(\frac{\pi}{2} - \theta) = \cos\phi.$$

In essence we want to rotate the vector $\vec{u}$ through $\pi/2$ radians counterclockwise, Suppose the vector we get this way is $\vec{u}'$.

Note that the angle between $\vec{u}'$ and $\vec{v}$ is ϕ, which is what we want. On the other hand, $\vec{u}$ and $\vec{u}'$ have the same length. It follows that the area of the parallelogram is

$$|\vec{u}||\vec{v}|\sin\theta = |\vec{u}'||\vec{v}|\cos\phi = \vec{u}' \cdot \vec{v}.$$

Question 2.2. *If $\vec{u} = \langle a_1, a_2\rangle$, then what is the vector $\vec{u}'$?*

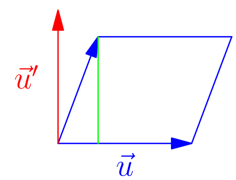

FIGURE 2. Rotated vector

Well by trial and error $\vec{u}' = \langle -a_2, a_1 \rangle$ is at right angles to $\vec{u}$ (the dot product is zero). The vector $\hat{\imath} = \langle 1, 0 \rangle$ gets sent to $\hat{\jmath} = \langle 0, 1 \rangle$, which is the right orientation (counterclockwise versus clockwise).

So the answer is $\vec{u}' = \langle -a_2, a_1 \rangle$.

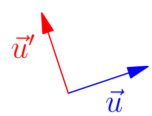

FIGURE 3. $\vec{u}$ and $\vec{u}'$

If $\vec{v} = \langle b_1, b_2 \rangle$, then putting all of this together, the formula for the area of the parallelogram is simply

$$a_1 b_2 - a_2 b_1.$$

Let

$$A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

be a 2×2 matrix. The determinant of A is

$$\det A = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1.$$

So the determinant of the matrix A is the $\pm$ the area of the parallelogram determined by the first and second row, $\vec{u} = \langle a_1, a_2 \rangle$ and $\vec{v} = \langle b_1, b_2 \rangle$. The sign depends on whether or not $\vec{u}$ comes before or after $\vec{v}$ (clockwise versus anticlockwise).

Now suppose we are given three vectors in $\mathbb{R}^3$,

$$\vec{u} = \langle a_1, a_2, a_3 \rangle \qquad \vec{v} = \langle b_1, b_2, b_3 \rangle \qquad \text{and} \qquad \vec{w} = \langle c_1, c_2, c_3 \rangle.$$

Put them into a 3×3 matrix, whose rows are the three vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$,

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

The determinant of A is

$$\det A = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

It is $\pm$ the volume of the parallelepiped determined by the three vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$. The sign of the determinant is determined by whether or not the three vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ form a right handed set.

Rule 2.3 (Right hand rule)**.** *The three vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ form a right handed set if when you point your right hand in the direction of $\vec{u}$, curl your fingers in the direction of $\vec{v}$ then your thumb points in the direction of $\vec{w}$.*

The cross product of two vectors $\vec{v}$ and $\vec{w}$ is the vector given by the formula

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \hat{\imath} \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \hat{k} \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

Geometrically, $\vec{v} \times \vec{w}$ is the vector whose length is the area of the parallelogram with sides $\vec{v}$ and $\vec{w}$ and whose direction is orthogonal to the plane spanned by $\vec{v}$ and $\vec{w}$, such that $\vec{v}$, $\vec{w}$ and $\vec{v} \times \vec{w}$ form a right handed set.

Question 2.4. *What is $\hat{\imath} \times \hat{\jmath}$?*

Here are the algebraic rules to manipulate the cross product:

(1) $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$.
(2) $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$.
(3) $(\lambda \vec{u}) \times \vec{v} = \lambda(\vec{u} \times \vec{v})$.

Note that $\vec{w} \times \vec{v} = -\vec{v} \times \vec{w}$. In particular $\vec{v} \times \vec{v} = \vec{0}$. One of the most useful features of the cross product is that the cross product $\vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$.

Question 2.5. *What is the equation of the plane containing $P_1 = (1, 1, 1)$, $P_2 = (1, 2, 3)$ and $P_3 = (-1, -2, 1)$?*

I will give you two methods to answer this question, the first using determinants the second using the cross product.

Two vectors in the plane are

$$\vec{v} = \overrightarrow{P_1P_2} = \langle 0, 1, 2 \rangle \qquad \text{and} \qquad \vec{w} = \overrightarrow{P_1P_3} = \langle -2, -3, 0 \rangle.$$

Let $P = (x, y, z)$ be a general point of space. P belongs to the plane if and only if the vector

$$\overrightarrow{P_1P} = \langle x - 1, y - 1, z - 1 \rangle,$$

lies in the plane. But this is the case if and only if the volume of the parallelepiped spanned by $\vec{v}$, $\vec{w}$ and $\langle x-1, y-1, z-1\rangle$ is zero, that is when

$$\begin{vmatrix} x-1 & y-1 & z-1 \\ 0 & 1 & 2 \\ -2 & -3 & 0 \end{vmatrix} = 0.$$

If we expand the determinant, we get

$$(x-1)\begin{vmatrix} 1 & 2 \\ -3 & 0 \end{vmatrix} - (y-1)\begin{vmatrix} 0 & 2 \\ -2 & 0 \end{vmatrix} + (z-1)\begin{vmatrix} 0 & 1 \\ -2 & -3 \end{vmatrix} = 6(x-1) - 4(y-1) + 2(z-1).$$

So

$$3(x-1) - 2(y-1) + (z-1) = 0,$$

and expanding we get

$$3x - 2y + z = 2.$$

Here is the second method. The plane is specified by fixing one point P_1 in the plane and requiring that every vector in the plane with tail P_1 is orthogonal to a fixed vector $\vec{n}$, a normal vector to the plane.

$\vec{n}$ is orthogonal to $\vec{v}$ and $\vec{w}$, so we could take $\vec{n}$ to be the cross product.

$$\vec{v}\times\vec{w} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & 2 \\ -2 & -3 & 0 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 1 & 2 \\ -3 & 0 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 0 & 2 \\ -2 & 0 \end{vmatrix} + \hat{k}\begin{vmatrix} 0 & 1 \\ -2 & -3 \end{vmatrix} = 6\hat{\imath} - 4\hat{\jmath} + 2\hat{k}.$$

Therefore $\vec{n} = 3\hat{\imath} - 2\hat{\jmath} + \hat{k}$ is a normal vector to the plane. P lies in the plane if and only if $\overrightarrow{P_1P}$ is orthogonal to $\vec{n}$, if and only if

$$\langle x-1, y-1, z-1\rangle \cdot \langle 3, -2, 1\rangle = 0.$$

Expanding gives the same equation as before.

There is one more product, which is sometimes useful. Given three vectors, $\vec{u}$, $\vec{v}$ and $\vec{w}$, the scalar triple product is the scalar

$$\vec{u}\cdot(\vec{v}\times\vec{w}).$$

It is the signed volume of the parallelepiped determined by $\vec{u}$, $\vec{v}$ and $\vec{w}$ (the sign is positive if $\vec{u}$, $\vec{v}$ and $\vec{w}$ form a right handed set and negative if they form a left handed set).