20. Line integrals

Let's look more at line integrals. Let's suppose we want to compute the line integral of $\vec{F} = y\hat{\imath} + x\hat{\jmath}$ around the curve C which is the sector of the unit circle whose angle is $\pi/4$, starting and ending at the origin. We break C into three curves,

$$C = C_1 + C_2 + C_3.$$

The line C_1 from $(0,0)$ to $(1,0)$, the arc C_2 of the unit circle starting at $(1,0)$ and ending at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and the line from this point back to the origin C_3.

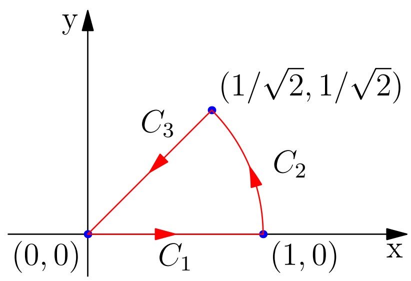

Figure 1. The curve C

We have

$$\int_C \vec{F}\cdot \mathrm{d}\vec{r} = \int_{C_1} \vec{F}\cdot \mathrm{d}\vec{r} + \int_{C_2} \vec{F}\cdot \mathrm{d}\vec{r} + \int_{C_3} \vec{F}\cdot \mathrm{d}\vec{r}.$$

We parametrise each curve separately.

The curve C_1: For the x-axis, $x(t) = t$, $y(t) = 0$, $0 \leq t \leq 1$. In this case

$$\vec{F} = \langle y, x\rangle = \langle 0, t\rangle \qquad \text{and} \qquad \mathrm{d}\vec{r} = \langle 1, 0\rangle\,\mathrm{d}t.$$

So

$$\int_{C_1} \vec{F}\cdot \mathrm{d}\vec{r} = \int_0^1 \langle 0, t\rangle \cdot \langle 1, 0\rangle\,\mathrm{d}t = \int_0^1 0\,\mathrm{d}t = 0.$$

In fact there are two other ways to see that we must get zero. We could take the arclength parametrisation. In this case $\hat{T} = \hat{\imath}$ and $\vec{F} = t\hat{\jmath}$, so that $\vec{F}\cdot\hat{T} = 0$. Or observe that the work done is zero, since the force is orthogonal to the velocity vector.

The curve C_2: For the arc of the circle, $x(t) = \cos t$, $y(t) = \sin t$, $0 \leq t \leq \pi/4$. In this case

$$\vec{F} = \langle y, x\rangle = \langle \sin t, \cos t\rangle \qquad \text{and} \qquad \mathrm{d}\vec{r} = \langle -\sin t, \cos t\rangle\,\mathrm{d}t.$$

So

$$\int_{C_2} \vec{F}\cdot \mathrm{d}\vec{r} = \int_0^{\pi/4} \langle \sin t, \cos t\rangle \cdot \langle -\sin t, \cos t\rangle \,\mathrm{d}t = \int_0^{\pi/4} \cos(2t)\,\mathrm{d}t = \left[\frac{\sin(2t)}{2}\right]_0^{\pi/4} = \frac{1}{2}.$$

The curve C_3: For the straight line segment starting at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and ending at the origin, we have $x(t) = t$, $y(t) = t$, $0 \leq t \leq 1/\sqrt{2}$.

$$\vec{F} = \langle y, x\rangle = \langle t, t\rangle \qquad \text{and} \qquad \mathrm{d}\vec{r} = \langle 1, 1\rangle \,\mathrm{d}t.$$

So,

$$\int_{C_3} \vec{F}\cdot \mathrm{d}\vec{r} = \int_{1/\sqrt{2}}^{0} \langle t, t\rangle \cdot \langle 1, 1\rangle \,\mathrm{d}t = \int_{1/\sqrt{2}}^{0} 2t\,\mathrm{d}t = \Big[t^2\Big]_{1/\sqrt{2}}^{0} = -\frac{1}{2}.$$

Note that the limits start at $1/\sqrt{2}$ and end at 0.

Putting all of this together, we get

$$\int_C \vec{F}\cdot \mathrm{d}\vec{r} = \int_{C_1} \vec{F}\cdot \mathrm{d}\vec{r} + \int_{C_2} \vec{F}\cdot \mathrm{d}\vec{r} + \int_{C_3} \vec{F}\cdot \mathrm{d}\vec{r} = 0 + 1/2 - 1/2 = 0.$$

We say that $\vec{F}$ is a gradient field if $\vec{F} = \nabla f$, for some scalar function f.

Theorem 20.1 (Fundamental Theorem of Calculus for line integrals)**.** *If $\vec{F} = \nabla f$ is a gradient vector field then*

$$\int_C \vec{F}\cdot \mathrm{d}\vec{r} = \int_C \nabla f \cdot \mathrm{d}\vec{r} = f(P_1) - f(P_0),$$

where C is a path from P_0 to P_1.

For example, suppose we take $f(x, y) = xy$. Then

$$\nabla f = y\hat{\imath} + x\hat{\jmath} = \vec{F},$$

the vector field above. Using (20.1), we see that

$$\int_C \vec{F}\cdot \mathrm{d}\vec{r} = f(0, 0) - f(0, 0) = 0.$$

On the other hand,

$$\int_{C_2} \vec{F}\cdot \mathrm{d}\vec{r} = f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) - f(1, 0) = \frac{1}{2}.$$

In the language of differentials, one can restate (20.1) as

$$\int_C f_x\,\mathrm{d}x + f_y\,\mathrm{d}y = \int_C \mathrm{d}f = f(P_1) - f(P_0).$$

Proof of (20.1).

$$\begin{aligned}\int_C \nabla f \cdot \mathrm{d}\vec{r} &= \int_{t_0}^{t_1} \left(f_x \frac{dx}{dt} + f_y \frac{dy}{dt} \right) \mathrm{d}t \\ &= \int_{t_0}^{t_1} \frac{d}{dt} \left(f(x(t), y(t)) \right) \mathrm{d}t \\ &= \Big[f(x(t), y(t)) \Big]_{t_0}^{t_1} \\ &= f(P_1) - f(P_0). \qquad \square\end{aligned}$$

(20.1) has some very interesting consequences:

Path independence: If C_1 and C_2 are two paths starting and ending at the same point, then

$$\int_{C_1} \nabla f \cdot \mathrm{d}\vec{r} = \int_{C_2} \nabla f \cdot \mathrm{d}\vec{r}.$$

In other words, the line integral

$$\int_C \nabla f \cdot \mathrm{d}\vec{r},$$

depends only on the endpoints, not on the trajectory.

Gradient fields are conservative: If C is a closed loop, then

$$\int_C \nabla f \cdot \mathrm{d}\vec{r} = 0.$$

We already saw that if C is a circle of radius a centred at the circle and $\vec{F} = -y\hat{\imath} + x\hat{\jmath}$, then

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = 2\pi a^2 \neq 0.$$

So the vector field $\vec{F} = -y\hat{\imath} + x\hat{\jmath}$ is not conservative. It follows that $\vec{F} = -y\hat{\imath} + x\hat{\jmath}$ is not the gradient of any scalar field.

If $\vec{F} = \nabla f$ is a gradient field, and $\vec{F}$ is the force, then f has an interesting physical interpretation, it is called the potential. In this case the work done is nothing more than the change in the potential. For example, if $\vec{F}$ is the force due to gravity, f is inversely proportional to the height. If $\vec{F}$ is the electric field, f is the voltage. (Note the annoying fact that mathematicians and physicists use a different sign convention; for physicists $\vec{F} = -\nabla f$).

To summarise, we have four equivalent properties:

(1) $\vec{F}$ is conservative, that is, $\int_C \vec{F} \cdot \mathrm{d}\vec{r} = 0$ for any closed loop.
(2) $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$ is path independent.

(3) $\vec{F} = \nabla f$ is a gradient vector field.

(4) $M\,\mathrm{d}x + N\,\mathrm{d}y$ is an exact differential, equal to $\mathrm{d}f$.

(1) and (2) are equivalent by considering the closed loop $C = C_1 - C_2$. (3) implies (2) by (20.1). We will see (2) implies (3) in the next lecture. (3) and (4) are the same statement, using different notation.