23. The flux

The flux of a vector field $\vec{F}$ across a curve C is

$$\int_C \vec{F} \cdot \hat{n} \, \mathrm{d}s,$$

where $\hat{n}$ is the unit normal vector to the curve C, obtained from the unit tangent vector $\hat{T}$ by rotating this vector through $\pi/2$ clockwise.

This gives us two line integrals:

We can integrate $\vec{F} \cdot \hat{T}$. In terms of Riemann sums, we add up the contributions from the component of $\vec{F}$ in the direction of $\hat{T}$, that is, along C. This computes the work done.

Or we can integrate $\vec{F} \cdot \hat{n}$. In terms of Riemann sums, we add up the contributions from the component of $\vec{F}$ in the direction of $\hat{n}$, that is, perpendicular to C. This computes the flux.

Suppose that $\vec{F}$ is a velocity vector field. Then the line integral

$$\int_C \vec{F} \cdot \hat{n} \, \mathrm{d}s,$$

represents how much matter crosses C in unit time.

To see this, let's fix ideas and suppose that $\vec{F}$ represents flow of water. Consider a small portion of C. Along this portion, $\vec{F}$ is approximately constant. The amount of water crossing C in unit time is given by a parallelogram with sides $\vec{F}$ and $(\Delta s)\hat{T}$. The area of this parallelogram is

$$(\vec{F} \cdot \hat{n})\Delta s;$$

$\vec{F} \cdot \hat{n}$ is the height of the parallelogram and Δs is the base. Dividing C into small pieces and summing all of these terms, gives a Riemann sum, an approximation to the total amount of water crossing C in unit time. Taking the limit as Δs goes to zero, the line integral

$$\int_C \vec{F} \cdot \hat{n} \, \mathrm{d}s,$$

represents how much water crosses C in unit time.

Note that water flowing left to right across C gets counted positively and water crossing right to left gets counted negatively (from the point of view of a particle travelling along C).

Example 23.1. *Suppose C is a circle of radius a, centre the origin. Let $\vec{F} = x\hat{\imath} + y\hat{\jmath}$. Then $\vec{F}$ points in the same direction as $\hat{n}$. So*

$$\vec{F} \cdot \hat{n} = |\vec{F}| = a.$$

It follows that

$$\oint_C \vec{F} \cdot \hat{n}\, \mathrm{d}s = \oint_C a\, \mathrm{d}s = 2\pi a^2.$$

The flux is $2\pi a^2$.

On the other hand, suppose we start with $\vec{F} = -y\hat{\imath} + x\hat{\jmath}$. *Then*

$$\vec{F} \cdot \hat{n} = 0,$$

since $\vec{F}$ *is perpendicular to* $\hat{n}$. *The flux is zero.*

Note that this makes sense physically. In the first example, water is spewing out of the origin. Lots of it crosses C. In the second example, water is spinning around the origin. None of it crosses C.

Now let's turn to how we would calculate the flux algebraically. We have

$$\mathrm{d}\vec{r} = \hat{T}\, \mathrm{d}s = \langle \mathrm{d}x, \mathrm{d}y \rangle.$$

The vector $\hat{n}$ is obtained from $\hat{T}$ by rotation through $\pi/2$ clockwise. So

$$\hat{n}\, \mathrm{d}s = \langle \mathrm{d}y, -\mathrm{d}x \rangle.$$

So as not to get lost in notation, let's suppose the components of $\vec{F}$ are P and Q,

$$\vec{F} = \langle P, Q \rangle = P\hat{\imath} + Q\hat{\jmath}.$$

We have

$$\int_C \vec{F} \cdot \hat{n}\, \mathrm{d}s = \int_C \langle P, Q \rangle \cdot \langle \mathrm{d}y, -\mathrm{d}x \rangle = \int_C -Q\, \mathrm{d}x + P\, \mathrm{d}y.$$

Theorem 23.2 (Green's Theorem for flux)**.** *If* C *is a positively oriented closed curve enclosing a region* R *and* $\vec{F} = P\hat{\imath} + Q\hat{\jmath}$ *then*

$$\oint_C -Q\, \mathrm{d}x + P\, \mathrm{d}y = \iint_R \operatorname{div} \vec{F}\, \mathrm{d}A, \qquad \textit{where} \qquad \operatorname{div} \vec{F} = P_x + Q_y.$$

Proof. Call $M = -Q$ and $N = P$, so that $\vec{G} = \langle M, N \rangle$ is $\vec{F}$ rotated through $\pi/2$ anticlockwise. Then we have

$$\begin{aligned}\oint_C -Q\,\mathrm{d}x + P\,\mathrm{d}y &= \oint_C M\,\mathrm{d}x + N\,\mathrm{d}y \\ &= \iint_R \operatorname{curl}\vec{G}\,\mathrm{d}y \\ &= \iint_R N_x - M_y\,\mathrm{d}y \\ &= \iint_R P_x + Q_y\,\mathrm{d}y \\ &= \iint_R \operatorname{div}\vec{F}\,\mathrm{d}A. \end{aligned}$$ □

Example 23.3. *Consider the example of a circle C of radius a, centre the origin. Suppose that $\vec{F} = x\hat{\imath} + y\hat{\jmath}$. Then*

$$\operatorname{div}\vec{F} = 1 + 1 = 2.$$

So the RHS of (23.2) *is*

$$\iint_R 2\,\mathrm{d}A = 2\pi a^2.$$

Suppose we move the circle away from the origin. Then computing the LHS becomes quite hard. But the RHS is unchanged.

$\operatorname{div}\vec{F}$ is called the divergence of $\vec{F}$. If $\vec{F}$ is the velocity vector field of water, the divergence measures how much water is being added (or taken away); these are known as sources (or sinks). For the vector field $\vec{F} = x\hat{\imath} + y\hat{\jmath}$, water is being added everywhere (imagine rain falling on the ground and then flowing away from the origin).

For the vector field $\vec{F} = -y\hat{\imath} + x\hat{\jmath}$ the divergence is zero. No water is being added or removed, there are no sources or sinks.