34. Review II

It is probably helpful to take stock of the various integrals and differentials we have encountered in this course:

Dimension	Standard	Vector
1	$\mathrm{d}t$, $\mathrm{d}s$	$\mathrm{d}\vec{r}$
2	$\mathrm{d}A$	$\mathrm{d}\vec{S}$
3	$\mathrm{d}V$	Not covered

In dimension one the most basic integral is the line integral:

$$\int_C \vec{F}\cdot \mathrm{d}\vec{r} = \int_C M\,\mathrm{d}x + N\,\mathrm{d}y.$$

This integral represents the work done to move a particle along C in a vector field $\vec{F}$. To compute directly, parametrise C. If we use the parameter t, we will get down to a standard one dimensional integral. For example, suppose that

$$\vec{F} = x\hat{\imath} + y\hat{\jmath}$$

and C is the unit circle, oriented counterclockwise. Paremetrise C in the standard way:

$$\vec{r}(t) = \langle \cos t, \sin t\rangle \qquad \text{where} \qquad 0 \leq t \leq 2\pi.$$

Then

$$\mathrm{d}\vec{r} = \langle -\sin t, \cos t\rangle\,\mathrm{d}t \qquad \text{and} \qquad \langle \cos t, \sin t\rangle.$$

Therefore

$$\oint_C \vec{F}\cdot \mathrm{d}\vec{r} = \int_0^{2\pi} 0\mathrm{d}t = 0.$$

One can also use Green's theorem. C bounds the unit disk R:

$$\oint_C \vec{F}\cdot \mathrm{d}\vec{r} = \iint_R \operatorname{curl}\vec{F}\,\mathrm{d}A = \int_0^{2\pi}\int_0^1 (0-0)r\,\mathrm{d}r\,\mathrm{d}\theta = 0,$$

as expected.

A closely related line integral is the flux of $\vec{F}$ across C. We measure the flux from left to right. The flux across C is

$$\int_C \vec{F}\cdot\hat{n}\,\mathrm{d}s.$$

To compute this, use the fact that $\hat{n}$ is the unit tangent vector turned through $\pi/2$ radians clockwise, so

$$\hat{n}\,\mathrm{d}s = \langle \mathrm{d}y, -\mathrm{d}x\rangle.$$

We have

$$\int_C \vec{F} \cdot \hat{n} \, \mathrm{d}s = \int_C M \mathrm{d}y - N \, \mathrm{d}x.$$

In the example above

$$\int_C \vec{F} \cdot \hat{n} \, \mathrm{d}s = \int_0^{2\pi} \cos^2 t + \sin^2 t \, \mathrm{d}t = 2\pi.$$

One can also use Green's theorem in normal form

$$\int_C \vec{F} \cdot \hat{n} \, \mathrm{d}s = \iint_R \operatorname{div} \vec{F} \, \mathrm{d}A = \iint_R 2 \, \mathrm{d}A = 2\pi.$$

$\mathrm{d}A$ is the area element in the xy-plane. We have

$$\mathrm{d}A = \mathrm{d}x \, \mathrm{d}y = r \, \mathrm{d}r \, \mathrm{d}\theta.$$

Example 34.1. *What is the area of the ellipse*

$$(2x + y)^2 + (x - y)^2 \leq 5?$$

Use change of variables, $u = 2x + y$ and $v = x - y$. The Jacobian is

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3.$$

So

$$\mathrm{d}u \, \mathrm{d}v = 3 \, \mathrm{d}x \, \mathrm{d}y.$$

So the area of R is

$$\iint_R 1 \, \mathrm{d}A = \iint_{(2x+y)^2+(x-y)^2 \leq 5} 1 \, \mathrm{d}x \, \mathrm{d}y = \iint_{u^2+v^2 \leq 5} \frac{1}{3} \, \mathrm{d}u \, \mathrm{d}v = \frac{5}{3}\pi.$$

Example 34.2. *Calculate*

$$\int_0^1 \int_{y^3}^1 \frac{6y^2}{x^2 + 2} \, \mathrm{d}x \, \mathrm{d}y.$$

We swap the order of integration. The region R of integration is

$$0 \leq y \leq 1 \qquad \text{and} \qquad y^3 \leq x \leq 1.$$

Therefore

$$\int_0^1 \int_{y^3}^1 \frac{6y^2}{x^2 + 2} \, \mathrm{d}x \, \mathrm{d}y = \iint_R \frac{6y^2}{x^2 + 2} \, \mathrm{d}x \, \mathrm{d}y = \int_0^1 \int_0^{x^{1/3}} \frac{6y^2}{x^2 + 2} \, \mathrm{d}y \, \mathrm{d}x.$$

The inner integral is

$$\int_0^{x^{1/3}} \frac{6y^2}{x^2 + 2} \, \mathrm{d}y = \left[\frac{2y^3}{x^2 + 2} \right]_0^{x^{1/3}} = \frac{2x}{2 + x^2}.$$

The outer integral is

$$\int_0^1 \frac{2x}{x^2+2}\,\mathrm{d}x = \Big[\ln(x^2+2)\Big]_0^1 = \ln 3/2.$$

In three dimensions, the volume form is

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z = \rho^2\sin\phi\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta.$$

The trickiest thing is to calculate surface integrals in space. The area form on a surface is

$$\mathrm{d}S.$$

It plays the same role as the area form $\mathrm{d}A$ in the plane. More common is

$$\mathrm{d}\vec{S} = \hat{n}\,\mathrm{d}S,$$

which is used to calculate flux out of S:

$$\iint_S \vec{F}\cdot\mathrm{d}\vec{S}.$$

Note that we need to choose an orientation of S. There are many ways to calculate the flux. If we parameterise S, $\vec{r}(u,v)$ using two parameters u and v we have

$$\mathrm{d}\vec{S} = \frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\,\mathrm{d}u\,\mathrm{d}v.$$

If S is given by a single constraint $g(x,y,z)=c$, a constant, then

$$\mathrm{d}\vec{S} = \frac{\vec{N}}{\vec{N}\cdot\hat{k}}\,\mathrm{d}x\,\mathrm{d}y \qquad \text{and} \qquad \mathrm{d}\vec{S} = \frac{\vec{N}}{|\vec{N}\cdot\hat{k}|}\,\mathrm{d}x\,\mathrm{d}y,$$

where $\vec{N} = \nabla g$ and the first form always picks the upwards orientation whilst the second form preserves the orientation. If S is given as the graph of a function $z = f(x,y)$ over a region R in the xy-plane, we have

$$\mathrm{d}\vec{S} = \langle -f_x, -f_y, 1\rangle\,\mathrm{d}x\,\mathrm{d}y.$$

Formulas for spheres centred at the origin and cylinders with central axis the z-axis are simply worth remembering:

$$\mathrm{d}\vec{S} = a\langle x,y,z\rangle\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta \qquad \text{and} \qquad \mathrm{d}\vec{S} = \langle x,y,0\rangle\,\mathrm{d}z\,\mathrm{d}\theta.$$

Example 34.3. *Let*

$$\vec{F} = \langle xz^2, yz^2, z^3\rangle.$$

What is the flux out of the cylinder, height 1, radius 1, base in the xy-plane, centred at the origin?

Let's calculate this directly. There are three sides, the two flat ones S_0 and S_1 and the curved one S_2.

For S_2, we have a cylinder, so use second form

$$\mathrm{d}\vec{S} = \langle x, y, 0 \rangle \, \mathrm{d}z \, \mathrm{d}\theta.$$

The flux across S_3 is

$$\iint_{S_2} x^2 z^2 + y^2 z^2 \, \mathrm{d}r \, \mathrm{d}\theta = \int_0^{2\pi} \int_0^1 z^2 \, \mathrm{d}z \, \mathrm{d}\theta.$$

The inner integral is

$$\int_0^1 z^2 \, \mathrm{d}z = \left[\frac{z^3}{3} \right]_0^1 = \frac{1}{3}.$$

So the flux across S_2 is $2\pi/3$. $\vec{F}$ is horizontal along S_0, so the flux across S_0 is zero. Across S_1, $\hat{n} = \hat{k}$, so the flux is

$$\iint_{S_1} \vec{F} \cdot \mathrm{d}\vec{S} = \iint_{S_1} 1 \mathrm{d}S = \pi,$$

since the area of S_1 is π.

In total, the flux is

$$\iint_S \vec{F} \cdot \mathrm{d}\vec{S} = \iint_{S_0} \vec{F} \cdot \mathrm{d}\vec{S} + \iint_{S_1} \vec{F} \cdot \mathrm{d}\vec{S} + \iint_{S_2} \vec{F} \cdot \mathrm{d}\vec{S} = 0 + \pi + \frac{2\pi}{3} = \frac{5\pi}{3}.$$

Instead we could apply the divergence theorem:

$$\iint_S \vec{F} \cdot \mathrm{d}\vec{S} = \iiint_V \operatorname{div} \vec{F} \, \mathrm{d}V = \iiint_V 5z^2 \, \mathrm{d}V.$$

To calculate this integral use cylindrical coordinates

$$\iiint_V 5z^2 \, \mathrm{d}V = \int_0^{2\pi} \int_0^1 \int_0^1 5z^2 r \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta = \frac{5\pi}{3}.$$

Here is a summary of the various fundamental theorems relating integrals in different dimensions:

Dimension	Work done	Flux
0-1	FTC line integrals	
1-2	Green's +Stokes' theorem	Green's theorem (normal form)
2-3	Divergence	Not covered