8. Review

Two ways to multiply vectors $\vec{v}$ and $\vec{w}$.

The dot product $\vec{v} \cdot \vec{w}$ takes two vectors and spits out a scalar, a number. Most important identity:

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\theta,$$

where θ is the angle between $\vec{v}$ and $\vec{w}$. Use this to compute θ.

Most important property:

$\vec{v}$ and $\vec{w}$ are orthogonal if and only if $\vec{v} \cdot \vec{w} = 0$.

Question 8.1. *What is the cosine of the angle between the vectors*

$$\vec{v} = \langle -1, 2, 2 \rangle \qquad \textit{and} \qquad \vec{w} = \langle 1, -4, 8 \rangle?$$

$$\cos\theta = \frac{\langle -1, 2, 2 \rangle \cdot \langle 1, -4, 8 \rangle}{|\langle -1, 2, 2 \rangle||\langle 1, -4, 8 \rangle|} = \frac{-1 - 8 + 16}{\sqrt{1+4+4}\sqrt{1+16+64}} = \frac{7}{27}.$$

The cross product $\vec{v} \times \vec{w}$ takes two vectors in $\mathbb{R}^3$ and spits out another vector in $\mathbb{R}^3$. Algebraically defined by determinants:

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Geometrically determined by:

magnitude of $\vec{v} \times \vec{w}$ is the area of the parallelogram given by $\vec{v}$ and $\vec{w}$, that is, $|\vec{v}||\vec{w}| \sin\theta$.

direction is determined by the following two properties:

(i) orthogonal to both $\vec{v}$ and $\vec{w}$,

(ii) the vectors $\vec{v}$, $\vec{w}$ and $\vec{v} \times \vec{w}$ form a right handed set.

Two important properties

$$\vec{v} \times \vec{w} = -\vec{w} \times \vec{v} \qquad \text{so that} \qquad \vec{v} \times \vec{v} = \vec{0}.$$

One can see the first property one of two ways. If you swap two rows of a determinant, the sign changes (the determinant is the signed volume of a parallelepiped). On the other hand as $\vec{v}$, $\vec{w}$ and $\vec{v} \times w$ are a right handed set, $\vec{w}$, $\vec{v}$ and $-\vec{v} \times \vec{w}$ are a right handed set.

Question 8.2. *What is the area of the triangle with sides*

$$\vec{v} = \langle -1, -2, 2 \rangle \qquad \textit{and} \qquad \langle 1, -2, 3 \rangle?$$

We want half the magnitude of the cross product. The cross product is

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & -2 & 2 \\ 1 & -2 & 3 \end{vmatrix} = \hat{\imath}\begin{vmatrix} -2 & 2 \\ -2 & 3 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} -1 & 2 \\ 1 & 3 \end{vmatrix} + \hat{k}\begin{vmatrix} -1 & -2 \\ 1 & -2 \end{vmatrix} = -2\hat{\imath} + 5\hat{\jmath} + 4\hat{k}.$$

Half the magnitude is

$$\frac{1}{2}(4 + 25 + 16)^{1/2} = \frac{1}{2}\sqrt{45} = \frac{3}{2}\sqrt{5}.$$

Planes in $\mathbb{R}^3$ are given by a single linear equation:

$$ax + by + cz = d.$$

Geometrically a plane is a determined by a point P_0 on the plane and a normal direction, $\vec{n}$. The point $P = (x, y, z)$ lies in the plane if and only if the vector $\overrightarrow{P_0P}$ is parallel to the plane, that is, if and only if

$$\overrightarrow{P_0P} \cdot \vec{n} = 0.$$

One can rewrite this equation as

$$\vec{P} \cdot \vec{n} = \vec{P}_0 \cdot \vec{n}.$$

If $\vec{n} = \langle 6, -2, -3 \rangle$, and $P_0 = (4, -1, 3)$ then

$$\langle x - 4, y + 1, z - 3 \rangle \cdot \langle 6, -2, -3 \rangle = 0,$$

that is

$$6(x - 4) - 2(y + 1) - 3(z - 3) = 0,$$

that is

$$6x - 2y - 3z = 31.$$

Note that one can read off a vector orthogonal to the plane from the equation immediately. The plane $ax + by + cz = d$ is orthogonal to $\vec{n} = \langle a, b, c \rangle$. d is a measure of how far the plane is from the origin; if $d = 0$ the plane passes through the origin. If d is not zero the plane has been translated in the direction of $\vec{n}$ (for example, consider horizontal planes, given by $z = 0$, $z = 1$, $z = 2$, $z = -1$, etc. They are planes translated up and down, that is, in the direction of $\hat{k}$.

How can one represent a line? One possibility is as the intersection of two planes. Each plane is determined by a single equation, so a line may be given to you as the set of solutions to two equations. For example, the solutions of the two equations

$$\begin{aligned} 2x - y + z &= 3 \\ 3x + y + z &= 1, \end{aligned}$$

represents a line.

Can one manipulate these two equations to get a single equation? **NO!**

This is important (because if you try to eliminate one equation, it is guaranteed you made a mistake and that you were wasting your time). There are lots of ways to see that this is not possible.

(1) We already decided that one equation represents a plane.

(2) Let's look at a concrete example. Suppose we start with the x-axis. Parametrically this is given as $\vec{r}(t) = t\hat{\imath} = \langle t, 0, 0\rangle$. How does one describe this by equations? Well $y = 0$ and $z = 0$ are two obvious equations. Clearly one cannot do better than this; no single linear equation will force both the component of $\hat{\jmath}$ and $\hat{k}$ to be zero.

(3) $\mathbb{R}^3$ is three dimensional. There are three degrees of freedom. Up-down, left-right, front-back. A plane has two degrees of freedom and a line one.

One equation imposes one condition, we lose one degree of freedom. So there are two degrees of freedom left. For example, the equation $y = 0$ means we can no longer go left-right, one constraint. We can still go up-down and front-back, so we still have two degrees of freedom. $y = 0$ represents a plane.

If we have two equations, each equation imposes one condition, so a pair of equations imposes two conditions. This leaves one degree of freedom. For example, $y = 0$ and $z = 0$ impose two conditions; you cannot move left-right and you cannot go up-down. This leaves one degree of freedom, front-back. The pair of equations $y = 0$ and $z = 0$ represents a line.

Question 8.3. *What is the equation of the plane containing the point* $P_0 = (3, -4, 1)$ *and the line given as the intersection of the two planes*

$$\begin{aligned} 2x - y + z &= 3 \\ 3x + y + z &= 1? \end{aligned}$$

We need to find the normal direction $\vec{n}$ of the plane. For this we need two vectors $\vec{v}$ and $\vec{w}$ parallel to the plane. For this we need two points P_1 and P_2 in the plane.

Obviously we want to choose two points P_1 and P_2 belonging to the line. Intersect the line with a plane to get a point. Take $x = 0$. Put this into the two equations we get

$$\begin{aligned} -y + z &= 3 \\ y + z &= 1. \end{aligned}$$

$z = 2$ and $y = -1$. So $P_1 = (0, -1, 2)$ is a point on the line.

Or we could take $x = 2$.

$$\begin{aligned} -y + z &= -1 \\ y + z &= -5. \end{aligned}$$

In this case $2z = -6$, $z = -3$ and so $y = -2$. So $P_2 = (2, -2, -3)$ is a point on the plane.

The vectors

$$\vec{v} = \overrightarrow{P_0P_1} = \langle -3, 3, 1\rangle \qquad \text{and} \qquad \vec{w} = \overrightarrow{P_0P_2} = \langle -1, 2, -4\rangle$$

are parallel to the plane. The cross-product is orthogonal to the plane:

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -3 & 3 & 1 \\ -1 & 2 & -4 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 3 & 1 \\ 2 & -4 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} -3 & 1 \\ -1 & -4 \end{vmatrix} + \hat{k}\begin{vmatrix} -3 & 3 \\ -1 & 2 \end{vmatrix} = -14\hat{\imath} - 13\hat{\jmath} - 3\hat{k}.$$

So the equation of the plane is

$$\langle x - 3, y + 4, z - 1\rangle \cdot \langle -14, -13, -3\rangle = 0,$$

so that

$$-14(x - 3) - 13(y + 4) - 3(z - 1) = 0.$$

Rearranging, we get

$$14x + 13y + 3z = -7.$$

There is another way to represent lines, we can parametrise a line. If Q_0 and Q_1 are two points in $\mathbb{R}^3$, then

$$\vec{r}(t) = \vec{Q}_0 + t\overrightarrow{Q_0Q_1}.$$

When $t = 0$, $\vec{r}(0) = Q_0$ and when $t = 1$, $\vec{r}(1) = Q_1$. Given a value for t, we get a point of the line. If we put $\overrightarrow{Q_0Q_1} = \vec{v}$, then we rewrite this parametrisation as

$$\vec{r}(t) = \vec{Q}_0 + t\overrightarrow{Q_0Q_1} = \vec{Q}_0 + t\vec{v}.$$

Here $\vec{v} = \overrightarrow{Q_0Q_1}$ is the velocity vector of the particle (at time $t = 0$, it is at Q_0 and time $t = 1$ at Q_1, or one could just differentiate).

If $Q_0 = (1, 2, 3)$ and $Q_1 = (2, -5, 2)$, then a parametrisation of the line through Q_0 and Q_1 is

$$\vec{r}(t) = \langle 1, 2, 3\rangle + t\langle 1, -7, -1\rangle = \langle 1 + t, 2 - 7t, 3 - t\rangle.$$

Question 8.4. *What is the shortest distance between the two lines*

$$\vec{r}_1(t) = \langle 6+2t, -1+t, 8+2t\rangle \qquad \textit{and} \qquad \vec{r}_2(t) = \langle 5-2t, -3+2t, 1+t\rangle?$$

We first check that these two lines are not parallel. The lines are parallel to

$$\vec{v} = \langle 2, 1, 2\rangle \qquad \text{and} \qquad \vec{w} = \langle -2, 2, 1\rangle.$$

$\vec{v}$ and $\vec{w}$ are not parallel ($\vec{w}$ is not a multiple of $\vec{v}$) so the lines are not parallel.

There are at least three different ways to solve this problem. All of them rely on the following basic observation. Suppose that P_1 and P_2 are the two closest points on either line. Then $\overrightarrow{P_0P_1}$ is orthogonal to both $\vec{v}$ and $\vec{w}$.

Now we describe the three methods. First the basic principle.

(1) Pick two random points R_1 and R_2 on the lines. The length of $\overrightarrow{P_1P_2}$ is nothing more than the (absolute value of the) component of $\overrightarrow{R_1R_2}$ in the direction of $\overrightarrow{P_1P_2}$.
(2) There are two parallel planes, containing either line. To find the distance between two parallel planes is relatively easy.
(3) $\overrightarrow{P_1P_2}$ is orthogonal to both $\vec{v}$ and $\vec{w}$. This gives two equations for the position of P_1 and P_2 and using this we can find P_1 and P_2.

Now to the execution.

Method #1: If we set $t = 0$ then we get two random points,

$$R_1 = (6, -1, 8) \qquad \text{and} \qquad R_2 = (5, -3, 1).$$

As the vector $\overrightarrow{P_1P_2}$ is orthogonal to $\vec{v}$ and $\vec{w}$, it is parallel to the cross product:

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 2 & 2 \\ -2 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix} = -3\hat{\imath} - 6\hat{\jmath} + 6\hat{k}.$$

So $\langle 1, 2, -2\rangle$ is othogonal to both $\vec{v}$ and $\vec{w}$. Dividing by the length, we get a unit vector orthogonal to both $\vec{v}$ and $\vec{w}$,

$$\hat{u} = \frac{1}{3}\langle 1, 2, -2\rangle.$$

The component of

$$\overrightarrow{R_1R_2} = \langle -1, -2, -7\rangle,$$

in the direction of $\hat{u}$, is

$$|\overrightarrow{R_1R_2}|\cos\theta.$$

But

$$\overrightarrow{R_1R_2} \cdot \hat{u} = |\overrightarrow{R_1R_2}||\hat{u}|\cos\theta = |\overrightarrow{R_1R_2}|\cos\theta.$$

So we just need to take the dot product:

$$\langle -1, -2, -7\rangle \cdot \frac{1}{3}\langle 1, 2, -2\rangle = \frac{1}{3}(-1 - 4 + 14) = 3.$$

This distance is 3.

Method #2: If $\mathcal{P}_1$ contains the first line and is parallel to the second line, it must be parallel to $\vec{v}$ and $\vec{w}$. So it must be orthogonal to $\vec{n} = \langle 1, 2, -2\rangle$, the cross product. The first plane has normal vector $\vec{n}$ and passes through $R_1 = (6, -1, 8)$. Hence

$$0 = \langle x - 6, y + 1, z - 8\rangle \cdot \langle 1, 2, -2\rangle = (x - 6) + 2(y + 6) - 2(z - 8).$$

Rearranging, we get

$$x + 2y - 2z = -12.$$

Similarly the second plane contains $R_2 = (5, -3, 1)$ and has normal vector $\vec{n}$,

$$0 = \langle x - 5, y - 3, z - 1\rangle \cdot \langle 1, 2, -2\rangle = (x - 5) + 2(y - 3) - 2(z - 1).$$

Rearranging, we get

$$x + 2y - 2z = -3.$$

Pick any point on the first plane. $P = (0, 0, 6)$ lies on the first plane. The line through this point parallel to $\vec{n}$ meets the second plane at a point Q whose distance from P is the distance between the two planes (whence the two lines).

The line through P parallel to $\vec{n}$ is given by

$$\vec{r}(t) = \langle 0, 0, 6\rangle + t\langle 1, 2, -2\rangle = \langle t, 2t, 6 - 2t\rangle.$$

This is on the second plane when

$$t + 4t - 12 + 4t = -3 \qquad \text{so that} \qquad t = 1.$$

The point $Q = (1, 2, 4)$. $\overrightarrow{PQ} = \langle 1, 2, -2\rangle$, which has length 3.

Method #3: We first parametrise the first line with a different parameter s.

$$\overrightarrow{P_1P_2} = \vec{r}_2(t) - \vec{r}_1(s) = \langle -1, -2, -7\rangle - s\langle 2, 1, 2\rangle + t\langle -2, 2, 1\rangle.$$

$\overrightarrow{P_1P_2}$ is orthogonal to $\vec{v}$ and $\vec{w}$ if and only if

$$\overrightarrow{P_1P_2} \cdot \vec{v} = 0 \qquad \text{and} \qquad \overrightarrow{P_1P_2} \cdot \vec{w} = 0.$$

This gives us two equations for s and t,

$$-9s = 18$$
$$9t = 9$$

Hence $s = -2$, $t = 1$. The vector

$$\overrightarrow{P_1P_2} = \langle -1, -2, -7 \rangle + 2\langle 2, 1, 2 \rangle + \langle -2, 2, 1 \rangle = \langle 1, 2, -2 \rangle.$$

This has length 3.