TABLE OF CONTENTS

MECHANICS

MECHANICS

MECHANICAL PROPERTIES OF BODIES

MECHANICAL PROPERTIES OF BODIES (Cont.)

FLYWHEELS

PENDULUMS

FRICTION

MECHANICS

Throughout the Mechanics section in this Handbook, both English and metric SI data and formulas are given to cover the requirements of working in either system of measurement. Except for the passage entitled *The Use of the Metric SI System in Mechanics Calculations* formulas and text relating exclusively to SI are given in bold face type.

Terms and Definitions

Definitions.—The science of mechanics deals with the effects of forces in causing or preventing motion. *Statics* is the branch of mechanics that deals with bodies in equilibrium, i.e., the forces acting on them cause them to remain at rest or to move with uniform velocity. *Dynamics* is the branch of mechanics that deals with bodies not in equilibrium, i.e., the forces acting on them cause them to move with non-uniform velocity. *Kinetics* is the branch of dynamics that deals with both the forces acting on bodies and the motions that they cause. *Kinematics* is the branch of dynamics that deals only with the motions of bodies without reference to the forces that cause them.

Definitions of certain terms and quantities as used in mechanics follow:

Force may be defined simply as a push or a pull; the push or pull may result from the force of contact between bodies or from a force, such as magnetism or gravitation, in which no direct contact takes place.

Matter is any substance that occupies space; gases, liquids, solids, electrons, atoms, molecules, etc., all fit this definition.

Inertia is the property of matter that causes it to resist any change in its motion or state of rest.

Mass is a measure of the inertia of a body.

Work, in mechanics, is the product of force times distance and is expressed by a combination of units of force and distance, as foot-pounds, inch-pounds, meter-kilograms, etc. **The metric SI unit of work is the joule, which is the work done when the point of application of a force of one newton is displaced through a distance of one meter in the direction of the force.**

Power, in mechanics, is the product of force times distance divided by time; it measures the performance of a given amount of work in a given time. It is the rate of doing work and as such is expressed in foot-pounds per minute, foot-pounds per second, kilogram-meters per second, etc. **The metric SI unit is the watt, which is one joule per second.**

Horsepower is the unit of power that has been adopted for engineering work. One horsepower is equal to 33,000 foot-pounds per minute or 550 foot-pounds per second. The *kilowatt*, used in electrical work, equals 1.34 horsepower; or 1 horsepower equals 0.746 kilowatt. **However, in the metric SI, the term horsepower is not used, and the basic unit of power is the watt. This unit, and the derived units milliwatt and kilowatt, for example, are the same as those used in electrical work.**

Torque or moment of a force is a measure of the tendency of the force to rotate the body upon which it acts about an axis. The magnitude of the moment due to a force acting in a plane perpendicular to some axis is obtained by multiplying the force by the perpendicular distance from the axis to the line of action of the force. (If the axis of rotation is not perpendicular to the plane of the force, then the components of the force in a plane perpendicular to the axis of rotation are used to find the resultant moment of the force by finding the moment of each component and adding these component moments algebraically.) Moment or torque is commonly expressed in pound-feet, pound-inches, kilogram-meters, etc. **The metric SI unit is the newton-meter (N · m).**

Velocity is the time-rate of change of distance and is expressed as distance divided by time, that is, feet per second, miles per hour, centimeters per second, meters per second, etc.

Acceleration is defined as the time-rate of change of velocity and is expressed as velocity divided by time or as distance divided by time squared, that is, in feet per second, per second or feet per second squared; inches per second, per second or inches per second squared; centimeters per second, per second or centimeters per second squared; etc. **The metric SI unit is the meter per second squared.**

Unit Systems.—In mechanics calculations, both *absolute* and *gravitational* systems of units are employed. The fundamental units in absolute systems are *length*, *time*, and *mass*, and from these units, the dimension of force is derived. Two absolute systems which have been in use for many years are the cgs (centimeter-gram-second) and the MKS (meter-kilogram-second) systems. Another system, known as MKSA (meter-kilogram-second-ampere), links the MKS system of units of mechanics with electro magnetic units.

The Conference General des Poids et Mesures (CGPM), which is the body responsible for all international matters concerning the metric system, adopted in 1954 a rationalized and coherent system of units based on the four MKSA units and including the kelvin as the unit of temperature, and the candela as the unit of luminous intensity. In 1960, the CGPM formally named this system the 'Systeme International d'Unites,' for which the abbreviation is SI in all languages. In 1971, the 14th CGPM adopted a seventh base unit, the mole, which is the unit of quantity ("amount of substance"). Further details of the SI are given in the Weights and Measures section, and its application in mechanics calculations, contrasted with the use of the English system, is considered on page 116.

The fundamental units in gravitational systems are *length*, *time*, and *force*, and from these units, the dimension of mass is derived. In the gravitational system most widely used in English measure countries, the units of length, time, and force are, respectively, the foot, the second, and the pound. The corresponding unit of mass, commonly called the *slug*, is equal to 1 pound second^2 per foot and is derived from the formula, $M = W \div g$ in which M = mass in slugs, W = weight in pounds, and g = acceleration due to gravity, commonly taken as 32.16 feet per second^2. A body that weighs 32.16 lbs. on the surface of the earth has, therefore, a mass of one slug.

Many engineering calculations utilize a system of units consisting of the inch, the second, and the pound. The corresponding units of mass are pounds second^2 per inch and the value of g is taken as 386 inches per second^2.

In a gravitational system that has been widely used in metric countries, the units of length, time, and force are, respectively, the meter, the second, and the kilogram. The corresponding units of mass are kilograms second^2 per meter and the value of g is taken as 9.81 meters per second^2.

Acceleration of Gravity *g* Used in Mechanics Formulas.—The acceleration of a freely falling body has been found to vary according to location on the earth's surface as well as with height, the value at the equator being 32.09 feet per second, per second while at the poles it is 32.26 ft/sec^2. In the United States it is customary to regard 32.16 as satisfactory for most practical purposes in engineering calculations.

Standard Pound Force: For use in defining the magnitude of a standard unit of force, known as the *pound force*, a fixed value of 32.1740 ft/sec^2, designated by the symbol g_0, has been adopted by international agreement. As a result of this agreement, whenever the term mass, M, appears in a mechanics formula and the substitution $M = W/g$ is made, use of the standard value g_0 = 32.1740 ft/sec^2 is implied although as stated previously, it is customary to use approximate values for g except where extreme accuracy is required.

American National Standard Letter Symbols for Mechanics and Time-Related Phenomena *ANSI/ASME Y10.3M-1984*

Acceleration, angular	α (alpha)	Height	*h*
Acceleration, due to gravity	*g*	Inertia, moment of	*I* or *J*
Acceleration, linear	*a*	Inertia, polar (area) moment of[a]	*J*
Amplitude[a]	*A*	Inertia, product (area) moment of[a]	I_{xy}
Angle	α (alpha)	Length	*L* or *l*
	β (beta)	Load per unit distance[a]	*q* or *w*
	γ (gamma)	Load, total[a]	*P* or W
	θ (theta)	Mass	*m*
	ϕ (phi)	Moment of force, including bending moment	*M*
	ψ (psi)	Neutral axis, distance to extreme fiber from[a]	*c*
Angle, solid	Ω (omega)	Period	*T*
Angular frequency	ω (omega)	Poisson's ratio	μ (mu) or ν (nu)
Angular momentum	*L*	Power	*P*
Angular velocity	ω (omega)	Pressure, normal force per unit area	*p*
Arc length	*s*	Radius	*r*
Area	*A*	Revolutions per unit of time	*n*
Axes, through any point[a]	*X-X*, *Y-Y*, or *Z-Z*	Second moment of area (second axial moment of area)	I_a
Bulk modulus	*K*	Second polar moment of area	I_p or *J*
Breadth (width)	*b*	Section modulus	*Z*
Coefficient of expansion, linear[a]	α (alpha)	Shear force in beam section[a]	*V*
Coefficient of friction	μ (mu)	Spring constant (load per unit deflection)[a]	*k*
Concentrated load (same as force)	*F*	Statical moment of any area about a given axis[a]	*Q*
Deflection of beam, max[a]	δ (delta)	Strain, normal	ε (epsilon)
Density	ρ (rho)	Strain, shear	γ (gamma)
Depth	*d*, δ (delta), or *t*	Stress, concentration factor[a]	*K*
Diameter	*D* or *d*	Stress, normal	σ (sigma)
Displacement[a]	*u*, *v*, *w*	Stress, shear	τ (tau)
Distance, linear[a]	*s*	Temperature, absolute[b]	*T*, or θ (theta)
Eccentricity of application of load[a]	*e*	Temperature[b]	*t*, or θ (theta)
Efficiency[a]	η (eta)	Thickness	*d*, δ (delta), or *t*
Elasticity, modulus of	*E*	Time	*t*
Elasticity, modulus of, in shear	*G*	Torque	*T*
Elongation, total[a]	δ (delta)	Velocity, linear	*v*
Energy, kinetic	E_k, *K*, *T*	Volume	*V*
Energy, potential	E_p, *V*, or Φ (phi)	Wavelength	λ (lambda)
Factor of safety[a]	*N*, or *n*	Weight	*W*
Force or load, concentrated	*F*	Weight per unit volume	γ (gamma)
Frequency	*f*	Work	*W*
Gyration, radius of[a]	*k*		

[a] Not specified in Standard
[b] Specified in ANSI Y10.4-1982 (R1988)

The Use of the Metric SI System in Mechanics Calculations.—The SI system is a development of the traditional metric system based on decimal arithmetic; fractions are avoided. For each physical quantity, units of different sizes are formed by multiplying or dividing a single base value by powers of 10. Thus, changes can be made very simply by adding zeros or shifting decimal points. For example, the meter is the basic unit of length; the kilometer is a multiple (1,000 meters); and the millimeter is a sub-multiple (one-thousandth of a meter).

In the older metric system, the simplicity of a series of units linked by powers of 10 is an advantage for plain quantities such as length, but this simplicity is lost as soon as more complex units are encountered. For example, in different branches of science and engineering, energy may appear as the erg, the calorie, the kilogram-meter, the liter-atmosphere, or the horsepower-hour. In contrast, the SI provides only one basic unit for each physical quantity, and universality is thus achieved.

There are seven base-units, and in mechanics calculations three are used, which are for the basic quantities of length, mass, and time, expressed as the meter (m), the kilogram (kg), and the second (s). The other four base-units are the ampere (A) for electric current, the kelvin (K) for thermodynamic temperature, the candela (cd) for luminous intensity, and the mole (mol) for amount of substance.

The SI is a coherent system. A system of units is said to be coherent if the product or quotient of any two unit quantities in the system is the unit of the resultant quantity. For example, in a coherent system in which the foot is a unit of length, the square foot is the unit of area, whereas the acre is not. Further details of the SI, and definitions of the units, are given in the section "*METRIC SYSTEMS OF MEASUREMENT*" at the end of the book.

Other physical quantities are derived from the base-units. For example, the unit of velocity is the meter per second (m/s), which is a combination of the base-units of length and time. The unit of acceleration is the meter per second squared (m/s^2). By applying Newton's second law of motion — force is proportional to mass multiplied by acceleration — the unit of force is obtained, which is the $kg \cdot m/s^2$. This unit is known as the newton, or N. Work, or force times distance, is the $kg \cdot m^2/s^2$, which is the joule, (1 joule = 1 newton-meter) and energy is also expressed in these terms. The abbreviation for joule is J. Power, or work per unit time, is the $kg \cdot m^2/s^3$, which is the watt (1 watt = 1 joule per second = 1 newton-meter per second). The abbreviation for watt is W.

The coherence of SI units has two important advantages. The first, that of uniqueness and therefore universality, has been explained. The second is that it greatly simplifies technical calculations. Equations representing physical principles can be applied without introducing such numbers as 550 in power calculations, which, in the English system of measurement have to be used to convert units. Thus conversion factors largely disappear from calculations carried out in SI units, with a great saving in time and labor.

Mass, weight, force, load: SI is an absolute system (see *Unit Systems* on page 114), and consequently it is necessary to make a clear distinction between mass and weight. The *mass* of a body is a measure of its inertia, whereas the weight of a body is the *force* exerted on it by gravity. In a fixed gravitational field, weight is directly proportional to mass, and the distinction between the two can be easily overlooked. However, if a body is moved to a different gravitational field, for example, that of the moon, its weight alters, but its mass remains unchanged. Since the gravitational field on earth varies from place to place by only a small amount, and weight is proportional to mass, it is practical to use the weight of unit mass as a unit of force, and this procedure is adopted in both the English and older metric systems of measurement. In common usage, they are given the same names, and we say that a mass of 1 pound has a weight of 1 pound. In the former case the pound is being used as a unit of mass, and in the latter case, as a unit of force. This procedure is convenient in some branches of engineering, but leads to confusion in others.

As mentioned earlier, Newton's second law of motion states that force is proportional to mass times acceleration. Because an unsupported body on the earth's surface falls with acceleration g (32 ft/s^2 approximately), the pound (force) is that force which will impart an acceleration of g ft/s^2 to a pound (mass). Similarly, the kilogram (force) is that force which will impart an acceleration of g (9.8 meters per second^2 approximately), to a mass of one kilogram. In the SI, the *newton* is that force which will impart unit acceleration (1 m/s^2) to a mass of one kilogram. It is therefore smaller than the kilogram (force) in the ratio 1:g (about 1:9.8). This fact has important consequences in engineering calculations. The factor g now disappears from a wide range of formulas in dynamics, but appears in many formulas in statics where it was formerly absent. It is however not quite the same g, for reasons which will now be explained.

In the article on page 154, the mass of a body is referred to as M, but it is immediately replaced in subsequent formulas by W/g, where W is the weight in pounds (force), which leads to familiar expressions such as $WV^2 / 2g$ for kinetic energy. In this treatment, the M which appears briefly is really expressed in terms of the slug (page 114), a unit normally used only in aeronautical engineering. In everyday engineers' language, weight and mass are regarded as synonymous and expressions such as $WV^2 / 2g$ are used without pondering the distinction. Nevertheless, on reflection it seems odd that g should appear in a formula which has nothing to do with gravity at all. In fact the g used here is not the true, local value of the acceleration due to gravity, but an arbitrary standard value which has been chosen as part of the definition of the pound (force) and is more properly designated g_o (page 114). Its function is not to indicate the strength of the local gravitational field, but to convert from one unit to another.

In the SI the unit of mass is the *kilogram*, and the unit of force (and therefore weight) is the *newton*.

The following are typical statements in dynamics expressed in SI units:

A force of R newtons acting on a mass of M kilograms produces an acceleration of R/M meters per second^2. The kinetic energy of a mass of M kg moving with velocity V m/s is $\frac{1}{2} MV^2$ kg (m/s)^2 or $\frac{1}{2}MV^2$ joules. The work done by a force of R newtons moving a distance L meters is RL Nm, or RL joules. If this work were converted entirely into kinetic energy we could write $RL = \frac{1}{2}MV^2$ and it is instructive to consider the units. Remembering that the N is the same as the kg · m/s^2, we have (kg · m/s)^2 × m = kg (m/s)^2, which is obviously correct. It will be noted that g does not appear anywhere in these statements.

In contrast, in many branches of engineering where the weight of a body is important, rather than its mass, using SI units, g does appear where formerly it was absent. Thus, if a rope hangs vertically supporting a mass of M kilograms the tension in the rope is Mg N. Here g is the acceleration due to gravity, and its units are m/s^2. The ordinary numerical value of 9.81 will be sufficiently accurate for most purposes on earth. The expression is still valid elsewhere, for example, on the moon, provided the proper value of g is used. The maximum tension the rope can safely withstand (and other similar properties) will also be specified in terms of the newton, so that direct comparison may be made with the tension predicted.

Words like load and weight have to be used with greater care. In everyday language we might say "a lift carries a load of five people of average weight 70 kg," but in precise technical language we say that if the average mass is 70 kg, then the average weight is 70g N, and the total load (that is force) on the lift is 350g N.

If the lift starts to rise with acceleration a m/s^2, the load becomes 350 $(g + a)$ N; both g and a have units of m/s^2, the mass is in kg, so the load is in terms of kg · m/s^2, which is the same as the newton.

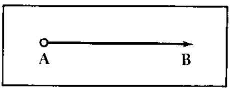
Pressure and stress: These quantities are expressed in terms of force per unit area. In the SI the unit is the pascal (Pa), which expressed in terms of SI derived and base units is the newton per meter squared (N/m^2). The pascal is very small—it is only equivalent to 0.15×10^{-3} lb/in^2 — hence the kilopascal (kPa = 1000 pascals), and the megapascal (MPa = 10^6 pascals) may be more convenient multiples in practice. Thus, note: 1 newton per millimeter squared = 1 meganewton per meter squared = 1 megapascal.

In addition to the pascal, the bar, a non-SI unit, is in use in the field of pressure measurement in some countries, including England. Thus, in view of existing practice, the International Committee of Weights and Measures (CIPM) decided in 1969 to retain this unit for a limited time for use with those of SI. The bar = 10^5 pascals and the hectobar = 10^7 pascals.

Force Systems

Scalar and Vector Quantities.—The quantities dealt with in mechanics are of two kinds according to whether magnitude alone or direction as well as magnitude must be known in order to completely specify them. Quantities such as time, volume and density are completely specified when their magnitude is known. Such quantities are called *scalar* quantities. Quantities such as force, velocity, acceleration, moment, and displacement which must, in order to be specified completely, have a specific direction as well as magnitude, are called *vector* quantities.

Graphical Representation of Forces.—A force has three characteristics which, when known, determine it. They are *direction, point of application,* and *magnitude*. The direction of a force is the direction in which it tends to move the body upon which it acts. The point of application is the place on the line of action where the force is applied. Forces may conveniently be represented by straight lines and arrow heads. The arrow head indicates the direction of the force, and the length of the line, its magnitude to any suitable scale. The point of application may be at any point on the line, but it is generally convenient to assume it to be at one end. In the accompanying illustration, a force is supposed to act along line *AB* in a direction from left to right. The length of line *AB* shows the magnitude of the force. If point *A* is the point of application, the force is exerted as a pull, but if point *B* be assumed to be the point of application, it would indicate that the force is exerted as a push.



Vector

Velocities, moments, displacements, etc. may similarly be represented and manipulated graphically because they are all of the same class of quantities called vectors. (See *Scalar and Vector Quantities*.)

Addition and Subtraction of Forces: The resultant of two forces applied at the same point and acting in the same direction, is equal to the sum of the forces. For example, if the two forces *AB* and *AC,* one equal to two and the other equal to three pounds, are applied at point *A*, then their resultant *AD* equals the sum of these forces, or five pounds.

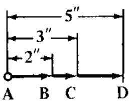


Fig. 1.

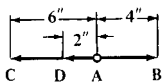


Fig. 2.

If two forces act in opposite directions, then their resultant is equal to their difference, and the direction of the resultant is the same as the direction of the greater of the two forces. For example: *AB* and *AC* are both applied at point *A*; then, if *AB* equals four and *AC* equals six pounds, the resultant *AD* equals two pounds and acts in the direction of *AC*.

Parallelogram of Forces: If two forces applied at a point are represented in magnitude and direction by the adjacent sides of a parallelogram (*AB* and *AC* in Fig. 3), their resultant will be represented in magnitude and direction by the diagonal *AR* drawn from the intersection of the two component forces.

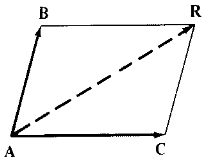


Fig. 3.

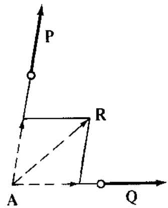


Fig. 4.

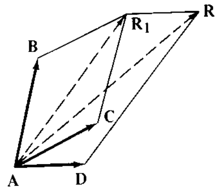


Fig. 5.

If two forces *P* and *Q* do not have the same point of application, as in Fig. 4, but the lines indicating their directions intersect, the forces may be imagined as applied at the point of intersection between the lines (as at *A*), and the resultant of the two forces may be found by constructing the parallelogram of forces. Line *AR* shows the direction and magnitude of the resultant, the point of application of which may be assumed to be at any point on line *AR* or its extension.

If the resultant of three or more forces having the same point of application is to be found, as in Fig. 5, first find the resultant of any two of the forces (*AB* and *AC*) and then find the resultant of the resultant just found (AR_1) and the third force (*AD*). If there are more than three forces, continue in this manner until the resultant of all the forces has been found.

Parallel Forces: If two forces are parallel and act in the same direction, as in Fig. 6, then their resultant is parallel to both lines, is located between them, and is equal to the sum of the two components. The point of application of the resultant divides the line joining the points of application of the components inversely as the magnitude of the components. Thus,

$$AB : CE = CD : AD$$

The resultant of two parallel and unequal forces acting in opposite directions, Fig. 7, is parallel to both lines, is located outside of them on the side of the greater of the components, has the same direction as the greater component, and is equal in magnitude to the difference between the two components. The point of application on the line *AC* produced is found from the proportion:

$$AB : CD = CE : AE$$

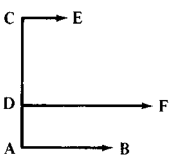


Fig. 6.

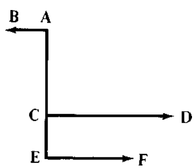


Fig. 7.

Polygon of Forces: When several forces are applied at a point and act in a single plane, Fig. 8, their resultant may be found more simply than by the method just described, as follows: From the extreme end of the line representing the first force, draw a line representing the second force, parallel to it and of the same length and in the direction of the second force. Then through the extreme end of this line draw a line parallel to, and of the same length and direction as the third force, and continue this until all the forces have been thus represented. Then draw a line from the point of application of the forces (as *A*) to the extreme point (as 5_1) of the line last drawn. This line (*A* 5_1) is the resultant of the forces.

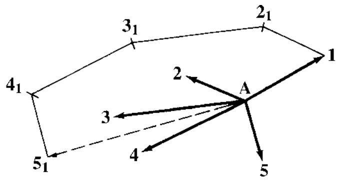


Fig. 8.

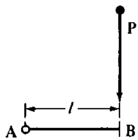
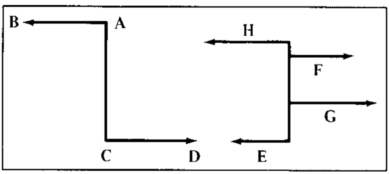


Fig. 9.

Moment of a Force: The moment of a force with respect to a point is the product of the force multiplied by the perpendicular distance from the given point to the direction of the force. In Fig. 9, the moment of the force *P* with relation to point *A* is $P \times AB$. The perpendicular distance *AB* is called the lever-arm of the force. The moment is the measure of the tendency of the force to produce rotation about the given point, which is termed the center of moments. If the force is measured in pounds and the distance in inches, the moment is expressed in inch-pounds. **In metric SI units, the moment is expressed in newton-meters (N · m), or newton-millimeters (N · mm).**

The moment of the resultant of any number of forces acting together in the same plane is equal to the algebraic sum of the moments of the separate forces.

Couples.—If the forces *AB* and *CD* are equal and parallel but act in opposite directions, then the resultant equals 0, or, in other words, the two forces have no resultant and are called a couple. A couple tends to produce rotation. The measure of this tendency is called the moment of the couple and is the product of one of the forces multiplied by the distance between the two.



Two Examples of Couples

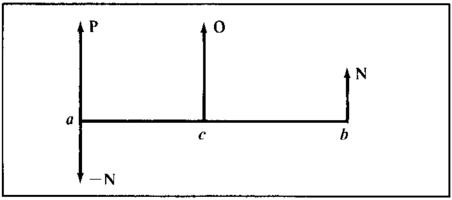
As a couple has no resultant, no single force can balance or counteract the tendency of the couple to produce rotation. To prevent the rotation of a body acted upon by a couple, two other forces are therefore required, forming a second couple. In the illustration, *E* and *F* form one couple and *G* and *H* are the balancing couple. The body on which they act is in equilibrium if the moments of the two couples are equal and tend to rotate the body in opposite directions. A couple may also be represented by a vector in the direction of the axis about which the couple acts. The length of the vector, to some scale, represents the magnitude of the couple, and the direction of the vector is that in which a right-hand screw would advance if it were to be rotated by the couple.

Composition of a Single Force and Couple.—A single force and a couple in the same plane or in parallel planes may be replaced by another single force equal and parallel to the first force, at a distance from it equal to the moment of the couple divided by the magnitude of the force. The new single force is located so that the moment of the resultant about the point of application of the original force is of the same sign as the moment of the couple.

In the next figure, with the couple $N-N$ in the position shown, the resultant of P, $-N$, and N is O (which equals P) acting on a line through point c so that $(P-N) \times ac = N \times bc$.

Thus, it follows that,

$$ac = \frac{N(ac+bc)}{P} = \frac{\text{Moment of Couple}}{P}$$



Single Force and Couple Composition

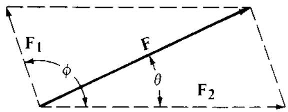
Algebraic Composition and Resolution of Force Systems.—The graphical methods given beginning on page 118 are convenient for solving problems involving force systems in which all of the forces lie in the same plane and only a few forces are involved. If many forces are involved, however, or the forces do not lie in the same plane, it is better to use algebraic methods to avoid complicated space diagrams. Systematic procedures for solving force problems by algebraic methods are outlined beginning on page 121. In connection with the use of these procedures, it is necessary to define several terms applicable to force systems in general.

The single force which produces the same effect upon a body as two or more forces acting together is called their *resultant*. The separate forces which can be so combined are called the *components*. Finding the resultant of two or more forces is called the *composition of forces*, and finding two or more components of a given force, the *resolution of forces*. Forces are said to be *concurrent* when their lines of action can be extended to meet at a common point; forces that are *parallel* are, of course, *nonconcurrent*. Two forces having the same line of action are said to be *collinear*. Two forces equal in magnitude, parallel, and in opposite directions constitute a *couple*. Forces all in the same plane are said to be *coplanar;* if not in the same plane, they are called *noncoplanar* forces.

The *resultant* of a system of forces is the simplest equivalent system that can be determined. It may be a single force, a couple, or a noncoplanar force and a couple. This last type of resultant, a noncoplanar force and a couple, may be replaced, if desired, by two *skewed* forces (forces that are nonconcurrent, nonparallel, and noncoplanar). When the resultant of a system of forces is zero, the system is in equilibrium, that is, the body on which the force system acts remains at rest or continues to move with uniform velocity.

Algebraic Solution of Force Systems—All Forces in the Same Plane

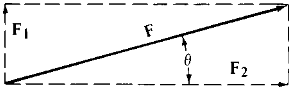
Finding Two Concurrent Components of a Single Force:



Case I: To find two components F_1 and F_2 at angles θ and φ, φ not being 90°.

$$F_1 = \frac{F \sin\theta}{\sin\phi}$$

$$F_2 = \frac{F \sin(\phi - \theta)}{\sin\phi}$$

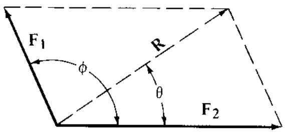


Case II: Components F_1 and F_2 form 90° angle.

$$F_1 = F \sin\theta$$

$$F_2 = F \cos\theta$$

Finding the Resultant of Two Concurrent Forces:

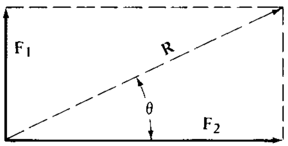


Case I: Forces F_1 and F_2 do not form 90° angle.

$$R = \frac{F_1 \sin\phi}{\sin\theta} \text{ or } R = \frac{F_2 \sin\phi}{\sin(\phi - \theta)} \text{ or}$$

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2 \cos\phi}$$

$$\tan\theta = \frac{F_1 \sin\phi}{F_1 \cos\phi + F_2}$$



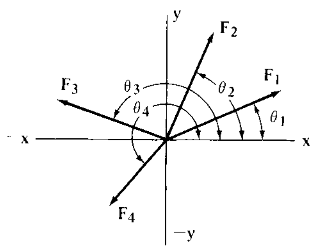
Case II: Forces F_1 and F_2 form 90° angle.

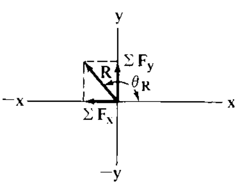
$$R = \frac{F_2}{\cos\theta} \quad \text{or} \quad R = \frac{F_1}{\sin\theta} \text{ or}$$

$$R = \sqrt{F_1{}^2 + F_2{}^2}$$

$$\tan\theta = \frac{F_1}{F_2}$$

Finding the Resultant of Three or More Concurrent Forces:





To determine resultant of forces F_1, F_2, F_3, etc. making angles, respectively, of θ_1, θ_2, θ_3, etc. with the *x* axis, find the *x* and *y* components F_x and F_y of each force and arrange in a table similar to that shown below for a system of three forces. Find the algebraic sum of the F_x and F_y components (ΣF_x and ΣF_y) and use these to determine resultant *R*.

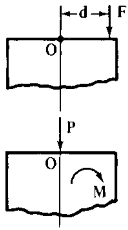
Force	F_x	F_y
F_1	$F_1 \cos \theta_1$	$F_1 \sin \theta_1$
F_2	$F_2 \cos \theta_2$	$F_2 \sin \theta_2$
F_3	$F_3 \cos \theta_3$	$F_3 \sin \theta_3$
	$\Sigma\Phi_x$	$\Sigma\Phi_y$

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$$

$$\cos\theta_R = \frac{\Sigma F_x}{R}$$

$$\text{or} \quad \tan\theta_R = \frac{\Sigma F_y}{\Sigma F_x}$$

Finding a Force and a Couple Which Together are Equivalent to a Single Force:

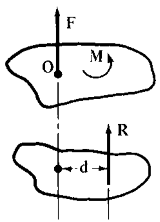


To resolve a single force F into a couple of moment M and a force P passing through any chosen point O at a distance d from the original force F, use the relations

$$P = F$$
$$M = F \times d$$

The moment M must, of course, tend to produce rotation about O in the same direction as the original force. Thus, as seen in the diagram, F tends to produce clockwise rotation; hence M is shown clockwise.

Finding the Resultant of a Single Force and a Couple:

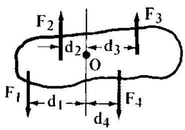


The resultant of a single force F and a couple M is a single force R equal in magnitude and direction to F and parallel to it at a distance d to the left or right of F.

$$R = F$$
$$d = M \div R$$

Resultant R is placed to the left or right of point of application O of the original force F depending on which position will give R the same direction of moment about O as the original couple M.

Finding the Resultant of a System of Parallel Forces:



To find the resultant of a system of coplanar parallel forces, proceed as indicated below.

1) Select any convenient point O from which perpendicular distances d_1, d_2, d_3, etc. to parallel forces F_1, F_2, F_3, etc. can be specified or calculated.

2) Find the algebraic sum of all the forces; this will give the magnitude of the resultant of the system.

$$R = \Sigma F = F_1 + F_2 + F_3 + \ldots$$

3) Find the algebraic sum of the moments of the forces about O; clockwise moments may be taken as negative and counterclockwise moments as positive:

$$\Sigma M_O = F_1 d_1 + F_2 d_2 + \ldots$$

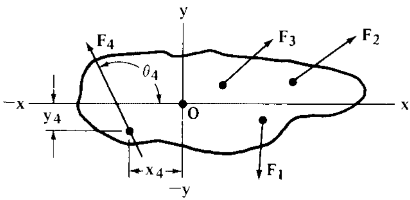
4) Calculate the distance d from O to the line of action of resultant R:

$$d = \Sigma M_O \div R$$

This distance is measured to the left or right from O depending on which position will give the moment of R the same direction of rotation about O as the couple ΣM_O, that is, if ΣM_O is negative, then d is left or right of O depending on which direction will make $R \times d$ negative.

Note Concerning Interpretation of Results: If $R = 0$, then the resultant of the system is a couple ΣM_O; if $\Sigma M_O = 0$ then the resultant is a single force R; if both R and $\Sigma M_O = 0$, then the system is in equilibrium.

Finding the Resultant of Forces Not Intersecting at a Common Point:



To determine the resultant of a coplanar, nonconcurrent, nonparallel force system as shown in the diagram, proceed as shown below.

1) Draw a set of *x* and *y* coordinate axes through any convenient point *O* in the plane of the forces as shown in the diagram.

2) Determine the *x* and *y* coordinates of any convenient point on the line of action of each force and the angle θ, measured in a counterclockwise direction, that each line of action makes with the positive *x* axis. For example, in the diagram, coordinates x_4, y_4, and θ_4 are shown for F_4. Similar data should be known for each of the forces of the system.

3) Calculate the *x* and *y* components (F_x, F_y) of each force and the moment of each component about *O*. Counterclockwise moments are considered positive and clockwise moments are negative. Tabulate all results in a manner similar to that shown below for a system of three forces and find ΣF_x, ΣF_y, ΣM_O by algebraic addition.

Force	Coordinates of *F*			Components of *F*		Moment of *F* about *O*
F	x	y	θ	F_x	F_y	$M_O = xF_y - yF_x$
F_1	x_1	y_1	θ_1	$F_1 \cos\theta_1$	$F_1 \sin\theta_1$	$x_1F_1 \sin\theta_1 - y_1F_1 \cos\theta_1$
F_2	x_2	y_2	θ_2	$F_2 \cos\theta_2$	$F_2 \sin\theta_2$	$x_2F_2 \sin\theta_2 - y_2F_2 \cos\theta_2$
F_3	x_3	y_3	θ_3	$F_3 \cos\theta_3$	$F_3 \sin\theta_3$	$x_3F_3 \sin\theta_3 - y_3F_3 \cos\theta_3$
				ΣF_x	ΣF_y	ΣM_O

4. Compute the resultant of the system and the angle θ_R it makes with the *x* axis by using the formulas:

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$$

$$\cos\theta_R = \Sigma F_x \div R \text{ or } \tan\theta_R = \Sigma F_y \div \Sigma F_x$$

5. Calculate the distance *d* from *O* to the line of action of the resultant *R*:

$$d = \Sigma M_O \div R$$

Distance *d* is in such direction from *O* as will make the moment of *R* about *O* have the same sign as ΣM_O.

Note Concerning Interpretation of Results: If $R = 0$, then the resultant is a couple ΣM_O; if $\Sigma M_O = 0$, then *R* passes through *O*; if both $R = 0$ and $\Sigma M_O = 0$, then the system is in equilibrium.

Example: Find the resultant of three coplanar nonconcurrent forces for which the following data are given.

$$F_1 = 10 \text{ lbs; } x_1 = 5 \text{ in.; } y_1 = -1 \text{ in.; } \theta_1 = 270°$$

$$F_2 = 20 \text{ lbs; } x_2 = 4 \text{ in.; } y_2 = 1.5 \text{ in.; } \theta_2 = 50°$$

$$F_3 = 30 \text{ lbs; } x_3 = 2 \text{ in.; } y_3 = 2 \text{ in.; } \theta_3 = 60°$$

$$F_{x_1} = 10\cos 270° = 10 \times 0 = 0 \text{ lbs.}$$

$$F_{x_2} = 20\cos 50° = 20 \times 0.64279 = 12.86 \text{ lbs.}$$

$$F_{x_3} = 30\cos 60° = 30 \times 0.5000 = 15.00 \text{ lbs.}$$

$$F_{y_1} = 10 \times \sin 270° = 10 \times (-1) = -10.00 \text{ lbs.}$$

$$F_{y_2} = 20 \times \sin 50° = 20 \times 0.76604 = 15.32 \text{ lbs.}$$

$$F_{y_3} = 30 \times \sin 60° = 30 \times 0.86603 = 25.98 \text{ lbs.}$$

$$M_{o_1} = 5 \times (-10) - (-1) \times 0 = -50 \text{ in. lbs.}$$

$$M_{o_2} = 4 \times 15.32 - 1.5 \times 12.86 = 41.99 \text{ in. lbs.}$$

$$M_{o_3} = 2 \times 25.98 - 2 \times 15 = 21.96 \text{ in. lbs.}$$

***Note:* When working in metric SI units, pounds are replaced by newtons (N); inches by meters or millimeters, and inch-pounds by newton-meters (N · m) or newton-millimeters (N · mm).**

Force F	Coordinates of F			Components of F		Moment of F about O
	x	y	θ	F_x	F_y	
$F_1 = 10$	5	–1	270°	0	–10.00	–50.00
$F_2 = 20$	4	1.5	50°	12.86	15.32	41.99
$F_3 = 30$	2	2	60°	15.00	25.98	21.96
				27.86	31.30	13.95

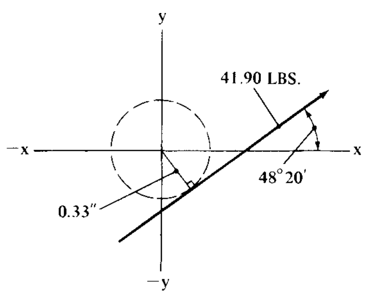
$$R = \sqrt{(27.86)^2 + (31.30)^2} = 41.90 \text{ lbs.}$$

$$\tan\theta_R = \frac{31.30}{27.86} = 1.1235$$

$$\theta_R = 48°20'$$

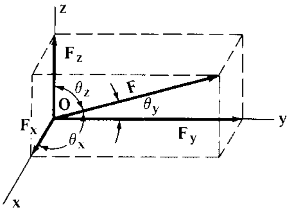
$$d = \frac{13.95}{41.90} = 0.33 \text{ inches}$$

measured as shown on the diagram.



Algebraic Solution of Force Systems — Forces Not in Same Plane

Resolving a Single Force Into Its Three Rectangular Components:

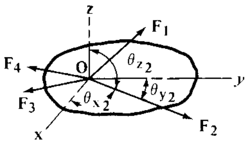


$$F_x = F\cos\theta_x$$
$$F_y = F\cos\theta_y$$
$$F_z = F\cos\theta_z$$
$$F = \sqrt{F_x^2 + F_y^2 + F_z^2}$$

The diagram shows how a force F may be resolved at any point O on its line of action into three concurrent components each of which is perpendicular to the other two.

The x, y, z components F_x, F_y, F_z of force F are determined from the accompanying relations in which $\theta_x, \theta_y, \theta_z$ are the angles that the force F makes with the x, y, z axes.

Finding the Resultant of Any Number of Concurrent Forces:



To find the resultant of any number of noncoplanar concurrent forces F_1, F_2, F_3, etc., use the procedure outlined below.

1) Draw a set of x, y, z axes at O, the point of concurrency of the forces. The angles each force makes measured counterclockwise from the positive x, y, and z coordinate axes must be known in addition to the magnitudes of the forces. For force F_2, for example, the angles are θ_{x2}, θ_{y2}, θ_{z2} as indicated on the diagram.

2) Apply the first three formulas given under the heading “Resolving a Single Force Into Its Three Rectangular Components” to each force to find its x, y, and z components. Tabulate these calculations as shown below for a system of three forces. Algebraically add the calculated components to find ΣF_x, ΣF_y, and ΣF_z which are the components of the resultant.

Force	Angles			Components of Forces		
F	θ_x	θ_y	θ_z	F_x	F_y	F_z
F_1	θ_{x1}	θ_{y1}	θ_{z1}	$F_1 \cos\theta_{x1}$	$F_1 \cos\theta_{y1}$	$F_1 \cos\theta_{z1}$
F_2	θ_{x2}	θ_{y2}	θ_{z2}	$F_2 \cos\theta_{x2}$	$F_2 \cos\theta_{y2}$	$F_2 \cos\theta_{z2}$
F_3	θ_{x3}	θ_{y3}	θ_{z3}	$F_3 \cos\theta_{x3}$	$F_3 \cos\theta_{y3}$	$F_3 \cos\theta_{z3}$
				ΣF_x	ΣF_y	ΣF_z

3. Find the resultant of the system from the formula $R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2 + (\Sigma F_z)^2}$

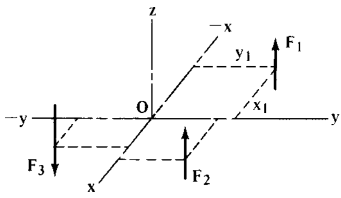
4. Calculate the angles θ_{xR}, θ_{yR}, and θ_{zR} that the resultant R makes with the respective coordinate axes:

$$\cos\theta_{xR} = \frac{\Sigma F_x}{R}$$

$$\cos\theta_{yR} = \frac{\Sigma F_y}{R}$$

$$\cos\theta_{zR} = \frac{\Sigma F_z}{R}$$

Finding the Resultant of Parallel Forces Not in the Same Plane:



In the diagram, forces F_1, F_2, etc. represent a system of noncoplanar parallel forces. To find the resultant of such systems, use the procedure shown below.

1) Draw a set of *x*, *y*, and *z* coordinate axes through any point *O* in such a way that one of these axes, say the *z* axis, is parallel to the lines of action of the forces. The *x* and *y* axes then will be perpendicular to the forces.

2) Set the distances of each force from the *x* and *y* axes in a table as shown below. For example, x_1 and y_1 are the *x* and *y* distances for F_1 shown in the diagram.

3) Calculate the moment of each force about the *x* and *y* axes and set the results in the table as shown for a system consisting of three forces. The algebraic sums of the moments ΣM_x and ΣM_y are then obtained. (In taking moments about the *x* and *y* axes, assign counterclockwise moments a plus (+) sign and clockwise moments a minus (-) sign. In deciding whether a moment is counterclockwise or clockwise, look from the positive side of the axis in question toward the negative side.)

Force	Coordinates of Force *F*		Moments M_x and M_y due to *F*	
F	x	y	M_x	M_y
F_1	x_1	y_1	F_1y_1	F_1x_1
F_2	x_2	y_2	F_2y_2	F_2x_2
F_3	x_3	y_3	F_3y_3	F_3x_3
ΣF			ΣM_x	ΣM_y

4. Find the algebraic sum ΣF of all the forces; this will be the resultant *R* of the system.

$$R = \Sigma F = F_1 + F_2 + \ldots$$

5. Calculate x_R and y_R, the moment arms of the resultant:

$$x_R = \Sigma M_y \div R$$

$$y_R = \Sigma M_x \div R$$

These moment arms are measured in such direction along the *x* and *y* axes as will give the resultant a moment of the same direction of rotation as ΣM_x and ΣM_y.

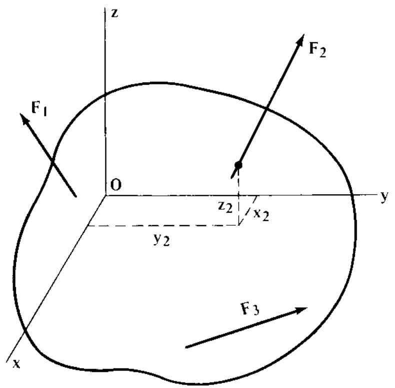
Note Concerning Interpretation of Results: If ΣM_x and ΣM_y are both 0, then the resultant is a single force *R* along the *z* axis; if *R* is also 0, then the system is in equilibrium. If *R* is 0 but ΣM_x and ΣM_y are not both 0, then the resultant is a couple

$$M_R = \sqrt{(\Sigma M_x)^2 + (\Sigma M_y)^2}$$

that lies in a plane parallel to the *z* axis and making an angle θ_R measured in a counterclockwise direction from the positive *x* axis and calculated from the following formula:

$$\sin\theta_R = \frac{\Sigma M_x}{M_R}$$

Finding the Resultant of Nonparallel Forces Not Meeting at a Common Point:



The diagram shows a system of noncoplanar, nonparallel, nonconcurrent forces F_1, F_2, etc. for which the resultant is to be determined. Generally speaking, the resultant will be a noncoplanar force and a couple which may be further combined, if desired, into two forces that are skewed.

This is the most general force system that can be devised, so each of the other systems so far described represents a special, simpler case of this general force system. The method of solution described below for a system of three forces applies for any number of forces.

1) Select a set of coordinate x, y, and z axes at any desired point O in the body as shown in the diagram.

2) Determine the x, y, and z coordinates of any convenient point on the line of action of each force as shown for F_2. Also determine the angles, θ_x, θ_y, θ_z that each force makes with each coordinate axis. These angles are measured counterclockwise from the positive direction of the x, y, and z axes. The data is tabulated, as shown in the table accompanying Step 3, for convenient use in subsequent calculations.

3) Calculate the x, y, and z components of each force using the formulas given in the accompanying table. Add these components algebraically to get ΣF_x, ΣF_y and ΣF_z which are the components of the resultant, R, given by the formula,

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2 + (\Sigma F_z)^2}$$

Force	Coordinates of Force F						Components of F		
F	x	y	z	θ_x	θ_y	θ_z	F_x	F_y	F_z
F_1	x_1	y_1	z_1	θ_{x1}	θ_{y1}	θ_{z1}	$F_1 \cos\theta_{x1}$	$F_1 \cos\theta_{y1}$	$F_1 \cos\theta_{z1}$
F_2	x_2	y_2	z_2	θ_{x2}	θ_{y2}	θ_{z2}	$F_2 \cos\theta_{x2}$	$F_2 \cos\theta_{y2}$	$F_2 \cos\theta_{z2}$
F_3	x_3	y_3	z_3	θ_{x3}	θ_{y3}	θ_{z3}	$F_3 \cos\theta_{x3}$	$F_3 \cos\theta_{y3}$	$F_3 \cos\theta_{z3}$
							ΣF_x	ΣF_y	ΣF_z

The resultant force R makes angles of θ_{xR}, θ_{yR}, and θ_{zR} with the x, y, and z axes, respectively, and passes through the selected point O. These angles are determined from the formulas,

$$\cos\theta_{xR} = \Sigma F_x \div R$$

$$\cos\theta_{yR} = \Sigma F_y \div R$$

$$\cos\theta_{zR} = \Sigma F_z \div R$$

4. Calculate the moments M_x, M_y, M_z about x, y, and z axes, respectively, due to the F_x, F_y, and F_z components of each force and set them in tabular form. The formulas to use are given in the accompanying table.

In interpreting moments about the x, y, and z axes, consider counterclockwise moments a plus (+) sign and clockwise moments a minus (-) sign. In deciding whether a moment is counterclockwise or clockwise, look from the positive side of the axis in question toward the negative side.

Force	Moments of Components of F (F_x, F_y, F_z) about x, y, z axes		
F	$M_x = yF_z - zF_y$	$M_y = zF_x - xF_z$	$M_z = xF_y - yF_x$
F_1	$M_{x1} = y_1F_{z1} - z_1F_{y1}$	$M_{y1} = z_1F_{x1} - x_1F_{z1}$	$M_{z1} = x_1F_{y1} - y_1F_{x1}$
F_2	$M_{x2} = y_2F_{z2} - z_2F_{y2}$	$M_{y2} = z_2F_{x2} - x_2F_{z2}$	$M_{z2} = x_2F_{y2} - y_2F_{x2}$
F_3	$M_{x3} = y_3F_{z3} - z_3F_{y3}$	$M_{y3} = z_3F_{x3} - x_3F_{z3}$	$M_{z3} = x_3F_{y3} - y_3F_{x3}$
	ΣM_x	ΣM_y	ΣM_z

5. Add the component moments algebraically to get ΣM_x, ΣM_y and ΣM_z which are the components of the resultant couple, M, given by the formula,

$$M = \sqrt{(\Sigma M_x)^2 + (\Sigma M_y)^2 + (\Sigma M_z)^2}$$

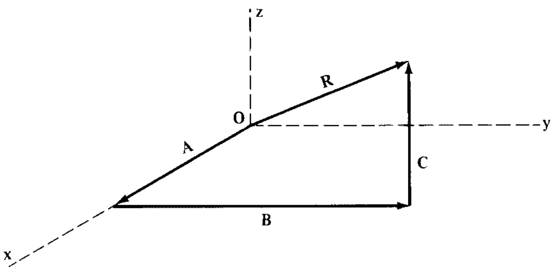
The resultant couple M will tend to produce rotation about an axis making angles of β_x, β_y, and β_z with the x, y, z axes, respectively. These angles are determined from the formulas,

$$\cos\beta_x = \frac{\Sigma M_x}{M} \qquad \cos\beta_y = \frac{\Sigma M_y}{M} \qquad \cos\beta_z = \frac{\Sigma M_z}{M}$$

General Method of Locating Resultant When Its Components are Known:

To determine the position of the resultant force of a system of forces, proceed as follows:

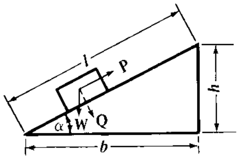
From the origin, point O, of a set of coordinate axes x, y, z, lay off on the x axis a length A representing the algebraic sum ΣF_x of the x components of all the forces. From the end of line A lay off a line B representing ΣF_y, the algebraic sum of the y components; this line B is drawn in a direction parallel to the y axis. From the end of line B lay off a line C representing ΣF_z. Finally, draw a line R from O to the end of C; R will be the resultant of the system.



Mechanisms

Inclined Plane—Wedge

W = weight of body



Neglecting friction:

$$P = W \times \frac{h}{l} = W \times \sin\alpha$$

$$W = P \times \frac{l}{h} = \frac{P}{\sin\alpha} = P \times \operatorname{cosec}\alpha$$

$$Q = W \times \frac{b}{l} = W \times \cos\alpha$$

If friction is taken into account, then force P to pull body up is:

$$P = W(\mu\cos\alpha + \sin\alpha)$$

Force P_1 to pull body down is:

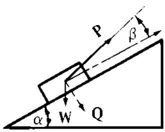
$$P_1 = W(\mu\cos\alpha - \sin\alpha)$$

Force P_2 to hold body stationary:

$$P_2 = W(\sin\alpha - \mu\cos\alpha)$$

in which μ is the coefficient of friction.

W = weight of body



Neglecting friction:

$$P = W \times \frac{\sin\alpha}{\cos\beta}$$

$$W = P \times \frac{\cos\beta}{\sin\alpha}$$

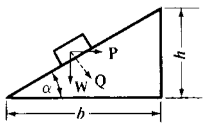
$$Q = W \times \frac{\cos(\alpha + \beta)}{\cos\beta}$$

With friction:

Coefficient of friction $= \mu = \tan\phi$

$$P = W \times \frac{\sin(\alpha + \phi)}{\cos(\beta - \phi)}$$

W = weight of body



Neglecting friction:

$$P = W \times \frac{h}{b} = W \times \tan\alpha$$

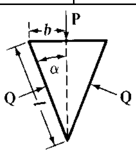
$$W = P \times \frac{b}{h} = P \times \cot\alpha$$

$$Q = \frac{W}{\cos\alpha} = W \times \sec\alpha$$

With friction:

Coefficient of friction $= \mu = \tan\phi$

$$P = W\tan(\alpha + \phi)$$



Neglecting friction:

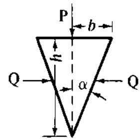
$$P = 2Q \times \frac{b}{l} = 2Q \times \sin\alpha$$

$$Q = P \times \frac{l}{2b} = \frac{1}{2}P \times \operatorname{cosec}\alpha$$

With friction:

Coefficient of friction = μ.

$$P = 2Q(\mu\cos\alpha + \sin\alpha)$$



Neglecting friction:

$$P = 2Q \times \frac{b}{h} = 2Q \times \tan\alpha$$

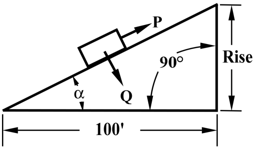
$$Q = P \times \frac{h}{2b} = \frac{1}{2}P \times \cot\alpha$$

With friction:

Coefficient of friction = $\mu = \tan\phi$.

$$P = 2Q\tan(\alpha + \phi)$$

Table of Forces on Inclined Planes



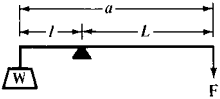
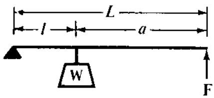
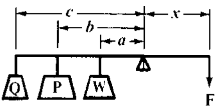
The table below makes it possible to find the force required for moving a body on an inclined plane. The friction on the plane is not taken into account. The column headed "Tension *P* in Cable per Ton of 2000 Pounds" gives the pull in pounds required for moving one ton along the inclined surface. The fourth column gives the perpendicular or normal pressure. If the coefficient of friction is known, the added pull required to overcome friction is thus easily determined:

$$Q \times \text{coefficient of friction} = \text{additional pull required.}$$

Tensions and Pressures in Pounds

Per Cent of Grade. Rise, Ft. per 100 Ft.	Angle α	Tension *P* in Cable per Ton of 2000 Lbs.	Perpendicular Pressure *Q* on Plane per Ton of 2000 Lbs.	Per Cent of Grade. Rise, Ft. Per 100 Ft.	Angle α	Tension *P* in Cable per Ton of 2000 Lbs.	Perpendicular Pressure *Q* on Plane per Ton of 2000 Lbs.
1	0° 35′	20.2	1999.8	39	21° 19′	727.0	1863.0
2	1 9	40.0	1999.4	40	21 49	743.2	1856.6
3	1 44	60.4	1999.0	41	22 18	758.8	1850.4
4	2 18	80.2	1998.2	42	22 47	774.4	1843.8
5	2 52	100.0	1997.4	43	23 17	790.4	1837.0
6	3 27	120.2	1996.2	44	23 45	805.4	1830.6
7	4 1	140.0	1995.0	45	24 14	820.8	1823.6
8	4 35	159.8	1993.6	46	24 43	836.2	1816.6
9	5 9	179.4	1991.8	47	25 11	851.0	1809.8
10	5 43	199.2	1990.0	48	25 39	865.6	1802.8
11	6 17	218.8	1987.8	49	26 7	880.4	1795.6
12	6 51	238.4	1985.6	50	26 34	894.4	1788.8
13	7 25	258.0	1983.2	51	27 2	909.0	1781.4
14	7 59	277.6	1980.6	52	27 29	922.8	1774.2
15	8 32	296.6	1977.8	53	27 56	936.8	1766.8
16	9 6	316.2	1974.8	54	28 23	950.6	1759.4
17	9 39	335.2	1971.6	55	28 49	964.0	1752.2
18	10 13	354.6	1968.2	56	29 15	977.2	1744.8
19	10 46	373.6	1964.6	57	29 41	990.4	1737.4
20	11 19	392.4	1961.0	58	30 7	1003.4	1730.0
21	11 52	411.2	1957.2	59	30 33	1016.4	1722.2
22	12 25	430.0	1953.2	60	30 58	1029.0	1714.8
23	12 58	448.6	1949.0	61	31 23	1041.4	1707.4
24	13 30	466.8	1944.6	62	31 48	1053.8	1699.6
25	14 3	485.4	1940.0	63	32 13	1066.2	1692.0
26	14 35	503.4	1935.4	64	32 38	1078.4	1684.2
27	15 7	521.4	1930.6	65	33 2	1090.2	1676.6
28	15 39	539.4	1925.8	66	33 26	1101.8	1669.0
29	16 11	557.4	1920.6	67	33 50	1113.4	1661.2
30	16 42	574.6	1915.6	68	34 13	1124.6	1653.8
31	17 14	592.4	1910.2	69	34 37	1136.0	1645.8
32	17 45	609.6	1904.6	70	35 0	1147.0	1638.2
33	18 16	626.8	1899.2	71	35 23	1158.0	1630.4
34	18 47	643.8	1893.4	72	35 46	1168.8	1622.8
35	19 18	661.0	1887.6	73	36 8	1179.2	1615.2
36	19 48	677.4	1881.6	74	36 31	1190.0	1607.2
37	20 19	694.4	1875.4	75	36 53	1200.4	1599.6
38	20 49	710.6	1869.4	…	… …	…	…

Levers

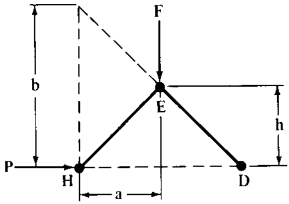
Types of Levers	Examples
 $F:W = l:L$ $\quad F \times L = W \times l$ $F = \frac{W \times l}{L}$ $\quad W = \frac{F \times L}{l}$ $L = \frac{W \times a}{W + F} = \frac{W \times l}{F}$ $\quad l = \frac{F \times a}{W + F} = \frac{F \times L}{W}$	A pull of 80 pounds is exerted at the end of the lever, at *W*; *l* = 12 inches and *L* = 32 inches. Find the value of force *F* required to balance the lever. $F = \frac{80 \times 12}{32} = \frac{960}{32} = 30 \text{ pounds}$ If *F* = 20; *W* = 180; and *l* = 3; how long must *L* be made to secure equilibrium? $L = \frac{180 \times 3}{20} = 27$
 $F:W = l:L$ $\quad F \times L = W \times l$ $F = \frac{W \times l}{L}$ $\quad W = \frac{F \times L}{l}$ $L = \frac{W \times a}{W - F} = \frac{W \times l}{F}$ $\quad l = \frac{F \times a}{W - F} = \frac{F \times L}{W}$	Total length *L* of a lever is 25 inches. A weight of 90 pounds is supported at *W*; *l* is 10 inches. Find the value of *F*. $F = \frac{90 \times 10}{25} = 36 \text{ pounds}$ If *F* = 100 pounds, *W* = 2200 pounds, and *a* = 5 feet, what should *L* equal to secure equilibrium? $L = \frac{2200 \times 5}{2200 - 100} = 5.24 \text{ feet}$
 When three or more forces act on lever: $F \times x = W \times a + P \times b + Q \times c$ $x = \frac{W \times a + P \times b + Q \times c}{F}$ $F = \frac{W \times a + P \times b + Q \times c}{x}$	Let *W* = 20, *P* = 30, and *Q* = 15 pounds; *a* = 4, *b* = 7, and *c* = 10 inches. If *x* = 6 inches, find *F*. $F = \frac{20 \times 4 + 30 \times 7 + 15 \times 10}{6} = 73\frac{1}{3} \text{ lbs}$ Assuming *F* = 20 in the example above, how long must lever arm *x* be made? $x = \frac{20 \times 4 + 30 \times 7 + 15 \times 10}{20} = 22 \text{ ins}$

The above formulas are valid using metric SI units, with forces expressed in newtons, and lengths in meters. However, it should be noted that the weight of a mass *W* kilograms is equal to a force of *Wg* newtons, where *g* is approximately 9.81 m/s^2. Thus, supposing that in the first example *l* = 0.4 m, *L* = 1.2 m, and *W* = 30 kg, then the weight of *W* is 30*g* newtons, so that the force *F* required to balance the lever is $F = \frac{30g \times 0.4}{1.2} = 10g = 98.1 \text{ newtons.}$

This force could be produced by suspending a mass of 10 kg at *F*.

Toggle-joint.—If arms *ED* and *EH* are of unequal length:

$$P = \frac{Fa}{b}$$



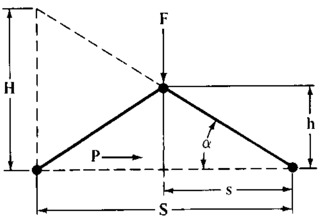
The relation between *P* and *F* changes constantly as *F* moves downward.

If arms *ED* and *EH* are equal:

$$P = \frac{Fa}{2h}$$

A double toggle-joint does not increase the pressure exerted so long as the relative distances moved by *F* and *P* remain the same.

Toggle-joints with Equal Arms



F = force applied
P = resistance
α = given angle

$$2P\sin\alpha = F\cos\alpha$$

$$\frac{P}{F} = \frac{\cos\alpha}{2\sin\alpha} = \text{coefficient}$$

$$P = F \times \text{coefficient}$$

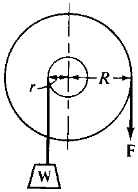
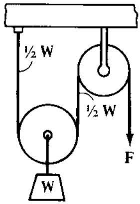
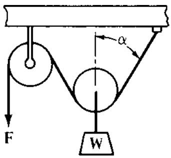
Equivalent expressions (see diagram):

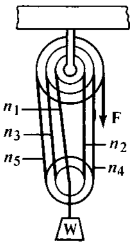
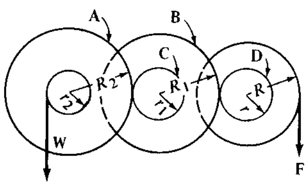
$$P = \frac{FS}{4h} \qquad P = \frac{Fs}{H}$$

To use the table, measure angle α, and find the coefficient in the table corresponding to the angle found. The coefficient is the ratio of the resistance to the force applied, and multiplying the force applied by the coefficient gives the resistance, neglecting friction.

Angle	Coefficient	Angle	Coefficient	Angle	Coefficient	Angle	Coefficient
0° 2′	862	0° 50′	34.4	2° 45′	10.4	8° 0′	3.58
0 4	456	0 55	31.2	2 50	10.1	8 30	3.35
0 6	285	1 0	28.6	3 0	9.54	9 0	3.15
0 8	216	1 10	24.6	3 15	8.81	9 30	2.99
0 10	171	1 15	22.9	3 30	8.17	10 0	2.84
0 12	143	1 20	21.5	3 45	7.63	11 0	2.57
0 14	122	1 30	19.1	4 0	7.25	12 0	2.35
0 15	115	1 40	17.2	4 15	6.73	13 0	2.17
0 16	107	1 45	16.4	4 30	6.35	14 0	2.00
0 18	95.4	1 50	15.6	4 45	6.02	15 0	1.87
0 20	85.8	2 0	14.3	5 0	5.71	16 0	1.74
0 25	68.6	2 10	13.2	5 30	5.19	17 0	1.64
0 30	57.3	2 15	12.7	6 0	4.76	18 0	1.54
0 35	49.1	2 20	12.5	6 30	4.39	19 0	1.45
0 40	42.8	2 30	11.5	7 0	4.07	20 0	1.37
0 45	38.2	2 40	10.7	7 30	3.79	…	…

Wheels and Pulleys

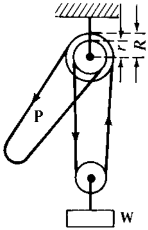
<table>
<tr>
<td>
</td>
<td>$F:W = r:R$
$F \times R = W \times r$
$F = \frac{W \times r}{R}$
$W = \frac{F \times R}{r}$
$R = \frac{W \times r}{F}$
$r = \frac{F \times R}{W}$</td>
<td>The radius of a drum on which is wound the lifting rope of a windlass is 2 inches. What force will be exerted at the periphery of a gear of 24 inches diameter, mounted on the same shaft as the drum and transmitting power to it, if one ton (2000 pounds) is to be lifted? Here $W = 2000$; $R = 12$; $r = 2$.
$F = \frac{2000 \times 2}{12} = 333$ pounds</td>
</tr>
<tr>
<td>
</td>
<td>$F = \frac{1}{2}W$
The velocity with which weight W will be raised equals one-half the velocity of the force applied at F.</td>
<td>

$F:W = \sec\alpha:2$
$F = \frac{W \times \sec\alpha}{2}$
$W = 2F \times \cos\alpha$</td>
</tr>
<tr>
<td>
</td>
<td>n = number of strands or parts of rope (n_1, n_2, etc.).
$F = \frac{1}{n} \times W$
The velocity with which W will be raised equals $\frac{1}{n}$ of the velocity of the force applied at F.</td>
<td>In the illustration is shown a combination of a double and triple block. The pulleys each turn freely on a pin as axis, and are drawn with different diameters, to show the parts of the rope more clearly. There are 5 parts of rope. Therefore, if 200 pounds is to be lifted, the force F required at the end of the rope is:
$F = \frac{1}{5} \times 200 = 40$ pounds</td>
</tr>
<tr>
<td colspan="2">

A, B, C and D are the pitch circles of gears.
$F = \frac{W \times r \times r_1 \times r_2}{R \times R_1 \times R_2}$
$W = \frac{F \times R \times R_1 \times R_2}{r \times r_1 \times r_2}$</td>
<td>Let the pitch diameters of gears A, B, C and D be 30, 28, 12 and 10 inches, respectively. Then $R_2 = 15$; $R_1 = 14$; $r_1 = 6$; and $r = 5$. Let $R = 12$, and $r_2 = 4$. Then the force F required to lift a weight W of 2000 pounds, friction being neglected, is:
$F = \frac{2000 \times 5 \times 6 \times 4}{12 \times 14 \times 15} = 95$ pounds</td>
</tr>
</table>

Note: The above formulas are valid using metric SI units, with forces expressed in newtons, and lengths in meters or millimeters. (See note on page 132 concerning weight and mass.)

Differential Pulley—Screw

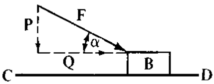


Differential Pulley.—In the differential pulley a chain must be used, engaging sprockets, so as to prevent the chain from slipping over the pulley faces.

$$P \times R = \tfrac{1}{2} W(R - r)$$

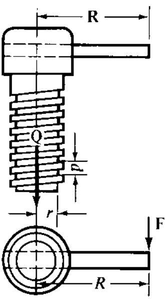
$$P = \frac{W(R - r)}{2R}$$

$$W = \frac{2PR}{R - r}$$



Force Moving Body on Horizontal Plane.—*F* tends to move *B* along line *CD*; *Q* is the component which actually moves *B*; *P* is the pressure, due to *F*, of the body on *CD*.

$$Q = F \times \cos\alpha \qquad P = \sqrt{F^2 - Q^2}$$



Screw.—*F* = force at end of handle or wrench; *R* = lever-arm of *F*; *r* = pitch radius of screw; *p* = lead of thread; *Q* = load. Then, neglecting friction:

$$F = Q \times \frac{p}{6.2832R} \qquad Q = F \times \frac{6.2832R}{p}$$

If μ is the coefficient of friction, then:

For motion in direction of load *Q* which *assists* it:

$$F = Q \times \frac{6.2832\mu r - p}{6.2832r + \mu p} \times \frac{r}{R}$$

For motion opposite load *Q* which *resists* it:

$$F = Q \times \frac{p + 6.2832\mu r}{6.2832r - \mu p} \times \frac{r}{R}$$

MECHANICAL PROPERTIES OF BODIES

Properties of Bodies

Center of Gravity.— The center of gravity of a body, volume, area, or line is that point at which if the body, volume, area, or line were suspended it would be perfectly balanced in all positions. For symmetrical bodies of uniform material it is at the geometric center. The center of gravity of a uniform round rod, for example, is at the center of its diameter halfway along its length; the center of gravity of a sphere is at the center of the sphere. For solids, areas, and arcs that are not symmetrical, the determination of the center of gravity may be made experimentally or may be calculated by the use of formulas.

The tables that follow give such formulas for some of the more important shapes. For more complicated and unsymmetrical shapes the methods outlined on page 142 may be used.

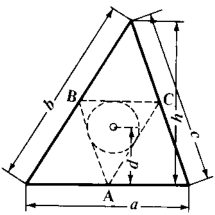
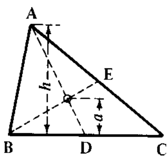
Example: A piece of wire is bent into the form of a semi-circular arc of 10-inch radius. How far from the center of the arc is the center of gravity located?

Accompanying the third diagram on page 137 is a formula for the distance from the center of gravity of an arc to the center of the arc: $a = 2r \div \pi$. Therefore,

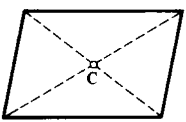
$$a = 2 \times 10 \div 3.1416 = 6.366 \text{ inches}$$

Formulas for Center of Gravity

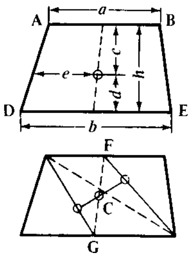
Triangle:

	*Perimeter* If *A*, *B* and *C* are the middle points of the sides of the triangle, then the center of gravity is at the center of the circle that can be inscribed in triangle *ABC*. The distance *d* of the center of gravity from side *a* is: $$d = \frac{h(b+c)}{2(a+b+c)}$$ where *h* is the height perpendicular to *a*.
	*Area* The center of gravity is at the intersection of lines *AD* and *BE*, which bisect the sides *BC* and *AC*. The perpendicular distance from the center of gravity to any one of the sides is equal to one-third the height perpendicular to that side. Hence, $a = h \div 3$.

Perimeter or Area of a Parallelogram :

	The center of gravity is at the intersection of the diagonals.

Area of Trapezoid:



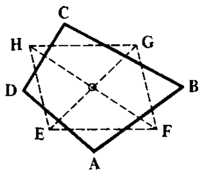
The center of gravity is on the line joining the middle points of parallel lines *AB* and *DE*.

$$c = \frac{h(a+2b)}{3(a+b)} \qquad d = \frac{h(2a+b)}{3(a+b)}$$

$$e = \frac{a^2+ab+b^2}{3(a+b)}$$

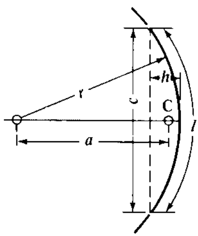
The trapezoid can also be divided into two triangles. The center of gravity is at the intersection of the line joining the centers of gravity of the triangles, and the middle line *FG*.

Any Four-sided Figure :



Two cases are possible, as shown in the illustration. To find the center of gravity of the four-sided figure *ABCD*, each of the sides is divided into three equal parts. A line is then drawn through each pair of division points next to the points of intersection *A*, *B*, *C*, and *D* of the sides of the figure. These lines form a parallelogram *EFGH*; the intersection of the diagonals *EG* and *FH* locates center of gravity.

Circular Arc:



The center of gravity is on the line that bisects the arc, at a distance $a = \frac{r \times c}{l} = \frac{c(c^2+4h^2)}{8lh}$ from the center of the circle.

For an arc equal to one-half the periphery:

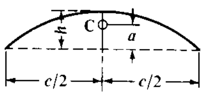
$$a = 2r \div \pi = 0.6366r$$

For an arc equal to one-quarter of the periphery:

$$a = 2r\sqrt{2} \div \pi = 0.9003r$$

For an arc equal to one-sixth of the periphery:

$$a = 3r \div \pi = 0.9549r$$

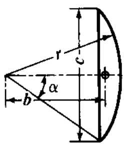


An approximate formula is very nearly exact for all arcs less than one-quarter of the periphery is:

$$a = \tfrac{2}{3}h$$

The error is only about one per cent for a quarter circle, and decreases for smaller arcs.

Circle Segment :

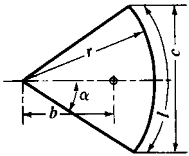


The distance of the center of gravity from the center of the circle is:

$$b = \frac{c^3}{12A} = \tfrac{2}{3} \times \frac{r^3 \sin^3\alpha}{A}$$

in which *A* = area of segment.

Circle Sector :



Distance b from center of gravity to center of circle is:

$$b = \frac{2rc}{3l} = \frac{r^2c}{3A} = 38.197\frac{r\sin\alpha}{\alpha}$$

in which A = area of sector, and α is expressed in degrees.

For the area of a half-circle:

$$b = 4r \div 3\pi = 0.4244r$$

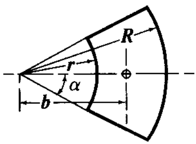
For the area of a quarter circle:

$$b = 4\sqrt{2} \times r \div 3\pi = 0.6002r$$

For the area of a sixth of a circle:

$$b = 2r \div \pi = 0.6366r$$

Part of Circle Ring :

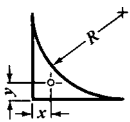


Distance b from center of gravity to center of circle is:

$$b = 38.197\frac{(R^3 - r^3)\sin\alpha}{(R^2 - r^2)\alpha}$$

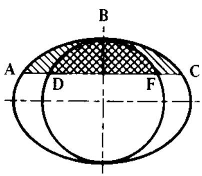
Angle α is expressed in degrees.

Spandrel or Fillet :



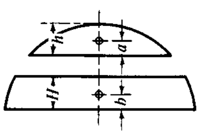
$$\text{Area} = 0.2146R^2 \qquad x = 0.2234R \qquad y = 0.2234R$$

Segment of an Ellipse :



The center of gravity of an elliptic segment *ABC*, symmetrical about one of the axes, coincides with the center of gravity of the segment *DBF* of a circle, the diameter of which is equal to that axis of the ellipse about which the elliptic segment is symmetrical.

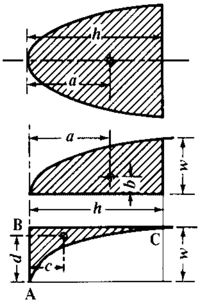
Spherical Surface of Segments and Zones of Spheres :



Distances a and b which determine the center of gravity, are:

$$a = \frac{h}{2} \qquad b = \frac{H}{2}$$

Area of a Parabola :



For the complete parabolic area, the center of gravity is on the center line or axis, and

$$a = \frac{3h}{5}$$

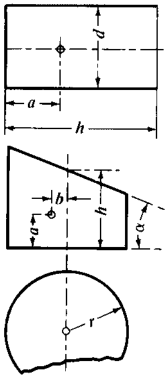
For one-half of the parabola:

$$a = \frac{3h}{5} \text{ and } b = \frac{3w}{8}$$

For the complement area *ABC*:

$$c = 0.3h \text{ and } d = 0.75w$$

Cylinder :



The center of gravity of a solid cylinder (or prism) with parallel end surfaces, is located at the middle of the line that joins the centers of gravity of the end surfaces.

The center of gravity of a cylindrical surface or shell, with the base or end surface in one end, is found from:

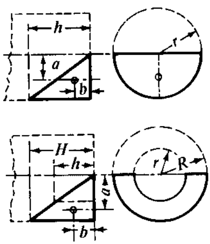
$$a = \frac{2h^2}{4h + d}$$

The center of gravity of a cylinder cut off by an inclined plane is located by:

$$a = \frac{h}{2} + \frac{r^2 \tan^2\alpha}{8h} \qquad b = \frac{r^2 \tan\alpha}{4h}$$

where α is the angle between the obliquely cut off surface and the base surface.

Portion of Cylinder :



For a solid portion of a cylinder, as shown, the center of gravity is determined by:

$$a = {}^3\!/_{16} \times 3.1416r \qquad b = {}^3\!/_{32} \times 3.1416h$$

For the cylindrical surface only:

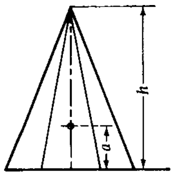
$$a = {}^1\!/_4 \times 3.1416r \qquad b = {}^1\!/_8 \times 3.1416h$$

If the cylinder is hollow, the center of gravity of the solid shell is found by:

$$a = {}^3\!/_{16} \times 3.1416\frac{R^4 - r^4}{R^3 - r^3}$$

$$b = {}^3\!/_{32} \times 3.1416\frac{H^4 - h^4}{H^3 - h^3}$$

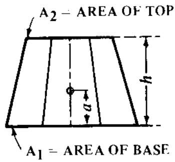
Pyramid :



In a solid pyramid the center of gravity is located on the line joining the apex with the center of gravity of the base surface, at a distance from the base equal to one-quarter of the height; or $a = \frac{1}{4}h$.

The center of gravity of the triangular surfaces forming the pyramid is located on the line joining the apex with the center of gravity of the base surface, at a distance from the base equal to one-third of the height; or $a = \frac{1}{3}h$.

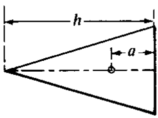
Frustum of Pyramid :



The center of gravity is located on the line that joins the centers of gravity of the end surfaces. If A_1 = area of base surface, and A_2 area of top surface,

$$a = \frac{h(A_1 + 2\sqrt{A_1 \times A_2} + 3A_2)}{4(A_1 + \sqrt{A_1 \times A_2} + A_2)}$$

Cone :



The same rules apply as for the pyramid.

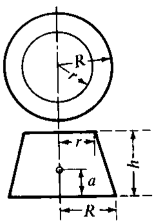
For the solid cone:

$$a = \tfrac{1}{4}h$$

For the conical surface:

$$a = \tfrac{1}{3}h$$

Frustum of Cone :



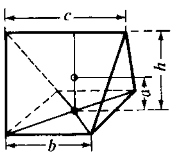
The same rules apply as for the frustum of a pyramid. For a solid frustum of a circular cone the formula below is also used:

$$a = \frac{h(R^2 + 2Rr + 3r^2)}{4(R^2 + Rr + r^2)}$$

The location of the center of gravity of the conical surface of a frustum of a cone is determined by:

$$a = \frac{h(R + 2r)}{3(R + r)}$$

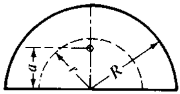
Wedge :



The center of gravity is on the line joining the center of gravity of the base with the middle point of the edge, and is located at:

$$a = \frac{h(b + c)}{2(2b + c)}$$

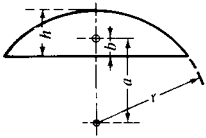
Half of a Hollow Sphere :



The center of gravity is located at:

$$a = \frac{3(R^4 - r^4)}{8(R^3 - r^3)}$$

Spherical Segment :



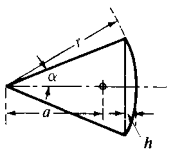
The center of gravity of a solid segment is determined by:

$$a = \frac{3(2r - h)^2}{4(3r - h)}$$

$$b = \frac{h(4r - h)}{4(3r - h)}$$

For a half-sphere, $a = b = \frac{3}{8}r$

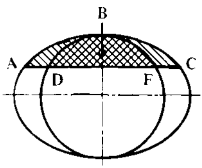
Spherical Sector :



The center of gravity of a solid sector is at:

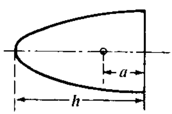
$$a = \tfrac{3}{8}(1 + \cos\alpha)r = \tfrac{3}{8}(2r - h)$$

Segment of Ellipsoid or Spheroid :



The center of gravity of a solid segment *ABC*, symmetrical about the axis of rotation, coincides with the center of gravity of the segment *DBF* of a sphere, the diameter of which is equal to the axis of rotation of the spheroid.

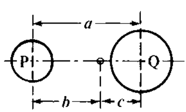
Paraboloid :



The center of gravity of a solid paraboloid of rotation is at:

$$a = \tfrac{1}{3}h$$

Center of Gravity of Two Bodies :



If the weights of the bodies are *P* and *Q*, and the distance between their centers of gravity is *a*, then:

$$b = \frac{Qa}{P + Q} \qquad c = \frac{Pa}{P + Q}$$

Center of Gravity of Figures of any Outline.—If the figure is symmetrical about a center line, as in Fig. 1, the center of gravity will be located on that line. To find the exact location on that line, the simplest method is by taking moments with reference to any convenient axis at right angles to this center line. Divide the area into geometrical figures, the centers of gravity of which can be easily found. In the example shown, divide the figure into three rectangles KLMN, EFGH and OPRS. Call the areas of these rectangles *A*, *B* and *C*, respectively, and find the center of gravity of each. Then select any convenient axis, as X–X, at right angles to the center line Y–Y, and determine distances *a*, *b* and *c*. The distance *y* of the center of gravity of the complete figure from the axis X–X is then found from the equation:

$$y = \frac{Aa + Bb + Cc}{A + B + C}$$

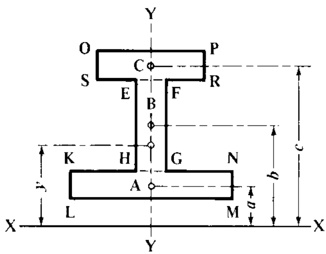


Fig. 1.

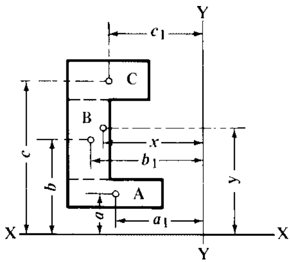


Fig. 2.

As an example, assume that the area *A* is 24 square inches, *B*, 14 square inches, and *C*, 16 square inches, and that *a* = 3 inches, *b* = 7.5 inches, and *c* = 12 inches. Then:

$$y = \frac{24 \times 3 + 14 \times 7.5 + 16 \times 12}{24 + 14 + 16} = \frac{369}{54} = 6.83 \text{ inches}$$

If the figure, the center of gravity of which is to be found, is not symmetrical about any axis, then moments must be taken with relation to two axes X–X and Y–Y, centers of gravity of which can be easily found, the same as before. The center of gravity is determined by the equations:

$$x = \frac{Aa_1 + Bb_1 + Cc_1}{A + B + C} \qquad y = \frac{Aa + Bb + Cc}{A + B + C}$$

As an example, let *A* = 14 square inches, *B* = 18 square inches, and *C* = 20 square inches. Let *a* = 3 inches, *b* = 7 inches, and *c* = 11.5 inches. Let a_1 = 6.5 inches, b_1 = 8.5 inches, and c_1 = 7 inches. Then:

$$x = \frac{14 \times 6.5 + 18 \times 8.5 + 20 \times 7}{14 + 18 + 20} = \frac{384}{52} = 7.38 \text{ inches}$$

$$y = \frac{14 \times 3 + 18 \times 7 + 20 \times 11.5}{14 + 18 + 20} = \frac{398}{52} = 7.65 \text{ inches}$$

In other words, the center of gravity is located at a distance of 7.65 inches from the axis X–X and 7.38 inches from the axis Y–Y.

Moments of Inertia.— An important property of areas and solid bodies is the moment of inertia. Standard formulas are derived by multiplying elementary particles of area or mass by the squares of their distances from reference axes. Moments of inertia, therefore, depend on the location of reference axes. Values are minimum when these axes pass through the centers of gravity.

Three kinds of moments of inertia occur in engineering formulas:

1) *Moments of inertia of plane area, I,* in which the axis is in the plane of the area, are found in formulas for calculating deflections and stresses in beams. When dimensions are given in inches, the units of I are inches^4. A table of formulas for calculating the I of common areas can be found in the *STRENGTH OF MATERIALS* section beginning on page 218.

2) *Polar moments of inertia of plane areas, J,* in which the axis is at right angles to the plane of the area, occur in formulas for the torsional strength of shafting. When dimensions are given in inches, the units of J are inches^4. If moments of inertia, I, are known for a plane area with respect to both x and y axes, then the polar moment for the z axis may be calculated using the equation, $J_z = I_x + I_y$

A table of formulas for calculating J for common areas can be found on page 278 in the *SHAFTS* section.

When metric SI units are used, the formulas referred to in (1) and (2) above, are valid if the dimensions are given consistently in meters or millimeters. If meters are used, the units of *I* and *J* are in meters^4; if millimeters are used, these units are in millimeters^4.

3) *Polar moments of inertia of masses,* J_M*, appear in dynamics equations involving rotational motion. J_M bears the same relationship to angular acceleration as mass does to linear acceleration. If units are in the foot-pound-second system, the units of J_M are ft-lbs-sec^2 or slug-ft^2. (1 slug = 1 pound second^2 per foot.) If units are in the inch-pound-second system, the units of J_M are inch-lbs-sec^2.

If metric SI values are used, the units of J_M are kilogram-meter squared. Formulas for calculating J_M for various bodies are given beginning on page 144. If the polar moment of inertia J is known for the area of a body of constant cross section, J_M may be calculated using the equation,

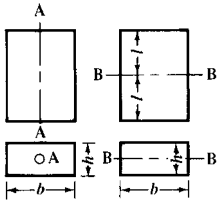
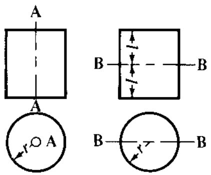
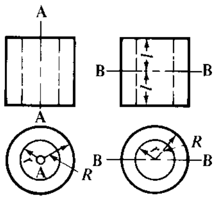
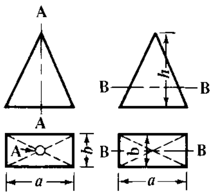
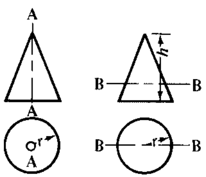
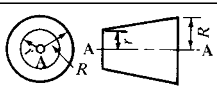
$$J_M = \frac{\rho L}{g} J$$

where ρ is the density of the material, L the length of the part, and g the gravitational constant. If dimensions are in the foot-pound-second system, ρ is in lbs per ft^3, L is in ft, g is 32.16 ft per sec^2, and J is in ft^4. If dimensions are in the inch-pound-second system, ρ is in lbs per in^3, L is in inches, g is 386 inches per sec^2, and J is in inches^4.

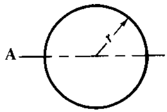
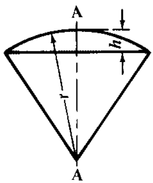
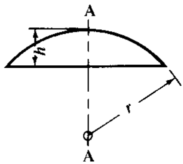
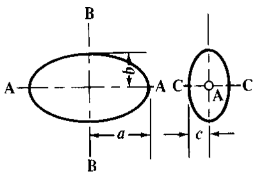
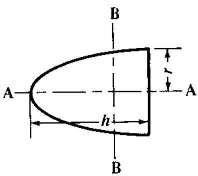
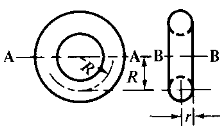
Using metric SI units, the above formula becomes $J_M = \rho LJ$, where ρ = the density in kilograms/meter^3, *L* = the length in meters, and *J* = the polar moment of inertia in meters^4. The units of J_M are $\text{kg} \cdot \text{m}^2$.

* In some books the symbol I denotes the polar moment of inertia of masses; J_M is used in this handbook to avoid confusion with moments of inertia of plane areas.

Formulas for Polar Moment of Inertia of Masses, J_M

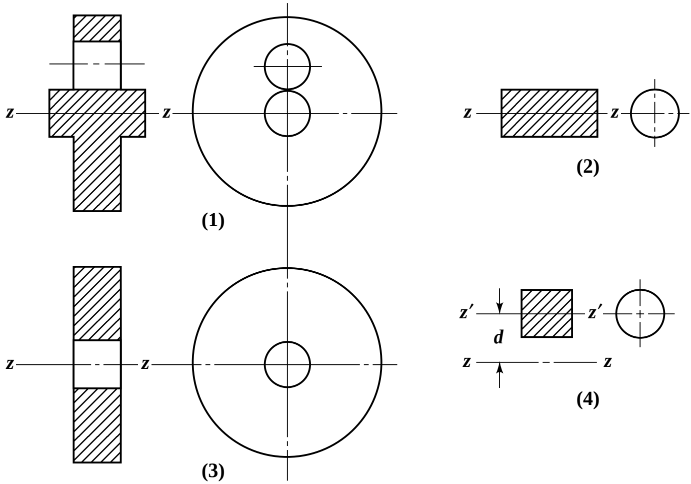
	*Prism:* With reference to axis $A – A$: $$J_M = \frac{M}{12}(h^2 + b^2)$$ With reference to axis $B – B$: $$J_M = M\left(\frac{l^2}{3} + \frac{h^2}{12}\right)$$
	*Cylinder*: With reference to axis $A – A$: $$J_M = \tfrac{1}{2}Mr^2$$ With reference to axis $B – B$: $$J_M = M\left(\frac{l^2}{3} + \frac{r^2}{4}\right)$$
	*Hollow Cylinder*: With reference to axis $A – A$: $$J_M = \tfrac{1}{2}M(R^2 + r^2)$$ With reference to axis $B – B$: $$J_M = M\left(\frac{l^2}{3} + \frac{R^2 + r^2}{4}\right)$$
	*Pyramid*, rectangular base: With reference to axis $A – A$: $$J_M = \frac{M}{20}(a^2 + b^2)$$ With reference to axis $B – B$ (through the center of gravity): $$J_M = M\left(\frac{3}{80}h^2 + \frac{b^2}{20}\right)$$
	*Cone:* With reference to axis $A – A$: $$J_M = \frac{3M}{10}r^2$$ With reference to axis $B – B$ (through the center of gravity): $$J_M = \frac{3M}{20}\left(r^2 + \frac{h^2}{4}\right)$$
	*Frustum of Cone:* With reference to axis $A – A$: $$J_M = \frac{3M(R^5 - r^5)}{10(R^3 - r^3)}$$

Formulas for Polar Moment of Inertia of Masses, J_M

	*Sphere*: With reference to any axis through the center: $$J_M = \tfrac{2}{5}Mr^2$$
	*Spherical Sector*: With reference to axis $A - A$: $$J_M = \frac{M}{5}(3rh - h^2)$$
	*Spherical Segment*: With reference to axis $A - A$: $$J_M = M\left(r^2 - \frac{3rh}{4} + \frac{3h^2}{20}\right)\frac{2h}{3r - h}$$
	*Ellipsoid*: With reference to axis $A - A$: $$J_M = \frac{M}{5}(b^2 + c^2)$$ With reference to axis $B - B$: $$J_M = \frac{M}{5}(a^2 + c^2)$$ With reference to axis $C - C$: $$J_M = \frac{M}{5}(a^2 + b^2)$$
	*Paraboloid*: With reference to axis $A - A$: $$J_M = \tfrac{1}{3}Mr^2$$ With reference to axis $B - B$ (through the center of gravity): $$J_M = M\left(\frac{r^2}{6} + \frac{h^2}{18}\right)$$
	*Torus*: With reference to axis $A - A$: $$J_M = M\left(\frac{R^2}{2} + \frac{5r^2}{8}\right)$$ With reference to axis $B - B$: $$J_M = M(R^2 + \tfrac{3}{4}r^2)$$

Moments of inertia of complex areas and masses may be evaluated by the addition and subtraction of elementary areas and masses. For example, the accompanying figure shows a complex mass at (1); its mass polar moment of inertia can be determined by adding together the moments of inertia of the bodies shown at (2) and (3), and subtracting that at (4).

Thus, $J_{M1} = J_{M2} + J_{M3} - J_{M4}$. All of these moments of inertia are with respect to the axis of rotation $z - z$. Formulas for J_{M2} and J_{M3} can be obtained from the tables beginning on page 144. The moment of inertia for the body at (4) can be evaluated by using the following transfer-axis equation: $J_{M4} = J_{M4}' + d^2M$. The term J_{M4}' is the moment of inertia with respect to axis $z' - z'$; it may be evaluated using the same equation that applies to J_{M2} where d is the distance between the $z - z$ and the $z' - z'$ axes, and M is the mass of the body (= weight in lbs ÷ g).



Moments of Inertia of Complex Masses

Similar calculations can be made when calculating I and J for complex areas using the appropriate transfer-axis equations are $I = I' + d^2A$ and $J = J' + d^2A$. The primed term, I' or J', is with respect to the center of gravity of the corresponding area A; d is the distance between the axis through the center of gravity and the axis to which I or J is referred.

Radius of Gyration.—The radius of gyration with reference to an axis is that distance from the axis at which the entire mass of a body may be considered as concentrated, the moment of inertia, meanwhile, remaining unchanged. If W is the weight of a body; J_M, its moment of inertia with respect to some axis; and k_o, the radius of gyration with respect to the same axis, then:

$$k_o = \sqrt{\frac{J_M g}{W}} \qquad \text{and} \qquad J_M = \frac{Wk_o^2}{g}$$

When using metric SI units, the formulas are:

$$\boldsymbol{k_o = \sqrt{\frac{J_M}{M}} \qquad \text{and} \qquad J_M = Mk_o^2}$$

where k_o = the radius of gyration in meters, J_M = kilogram-meter squared, and M = mass in kilograms.

To find the radius of gyration of an area, such as for the cross-section of a beam, divide the moment of inertia of the area by the area and extract the square root.

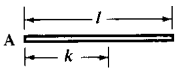
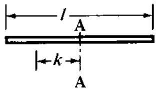
When the axis, the reference to which the radius of gyration is taken, passes through the center of gravity, the radius of gyration is the least possible and is called the *principal* radius of gyration. If k is the radius of gyration with respect to such an axis passing through the center of gravity of a body, then the radius of gyration, k_o, with respect to a parallel axis at a distance d from the gravity axis is given by:

$$k_o = \sqrt{k^2 + d^2}$$

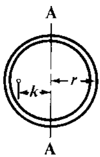
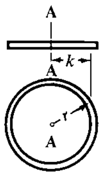
Tables of radii of gyration for various bodies and axes follows.

Formulas for Radius of Gyration

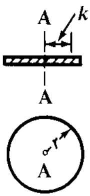
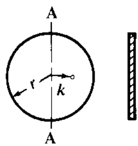
Bar of Small Diameter:

$k = 0.5773l$ $k^2 = \frac{1}{3}l^2$  Axis at end	$k = 0.2886l$ $K^2 = \frac{1}{12}l^2$  Axis at center

Bar of Small Diameter bent to Circular Shape:

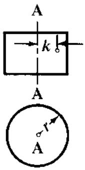
$k = 0.7071r$ $k^2 = \frac{1}{2}r^2$  Axis, a diameter of the ring	$k = r$; $k^2 = r^2$  Axis through center of ring.

Thin Circular Disk:

$k = 0.7071r$ $k^2 = \frac{1}{2}r^2$  Axis through center.	$k = \frac{1}{2}r$ $k^2 = \frac{1}{4}r^2$  Axis its diameter.

Cylinder:

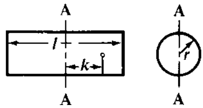
$k = 0.7071r, \qquad k^2 = \frac{1}{2}r^2$



Axis through center.

$k = 0.289\sqrt{l^2 + 3r^2}$

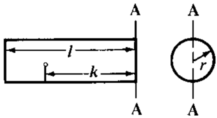
$k^2 = \frac{l^2}{12} + \frac{r^2}{4}$



Axis, diameter at mid-length.

$k = 0.289\sqrt{4l^2 + 3r^2}$

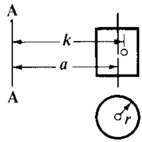
$k^2 = \frac{l^2}{3} + \frac{r^2}{4}$



Axis, diameter at end.

$k = \sqrt{a^2 + \frac{1}{2}r^2}$

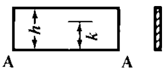
$k^2 = a^2 + \frac{1}{2}r^2$



Axis at a distance.

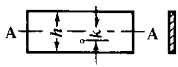
Parallelogram (Thin flat plate):

$k = 0.5773h$; $k^2 = \frac{1}{3}h^2$



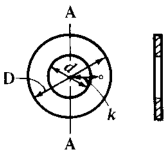
Axis at base.

$k = 0.2886h$; $k^2\ \frac{1}{12}h^2$



Axis at mid-height.

Thin, Flat, Circular Ring:

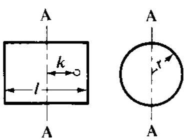


Axis its diameter.

$k = \frac{1}{4}\sqrt{D^2 + d^2}$

$k^2 = \frac{D^2 + d^2}{16}$

Thin Hollow Cylinder.:



Axis, diameter at mid-length.

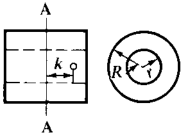
$k = 0.289\sqrt{l^2 + 6r^2}$

$k^2 = \frac{l^2}{12} + \frac{r^2}{2}$

Hollow Cylinder.:

$$k = 0.289\sqrt{l^2 + 3(R^2 + r^2)}$$

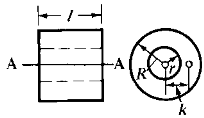
$$k^2 = \frac{l^2}{12} + \frac{R^2 + r^2}{4}$$



Axis, diameter at mid-length.

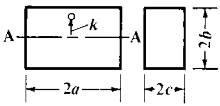
$$k = 0.7071\sqrt{R^2 + r^2}$$

$$k^2 = \tfrac{1}{2}(R^2 + r^2)$$



Longitudinal Axis.

Rectangular Prism:



Axis through center.

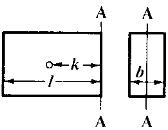
$$k = 0.577\sqrt{b^2 + c^2}$$

$$k^2 = \tfrac{1}{3}(b^2 + c^2)$$

Parallelepiped:

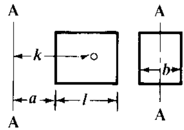
$$k = 0.289\sqrt{4l^2 + b^2}$$

$$k^2 = \frac{4l^2 + b^2}{12}$$



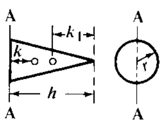
Axis at one end, central.

$$k = \sqrt{\frac{4l^2 + b^2}{12} + a^2 + al}$$



Axis at distance from end.

Cone:

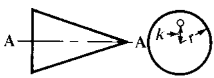


Axis at base.

$$k = \sqrt{\frac{2h^2 + 3r^2}{20}}$$

Axis at apex.

$$k_1 = \sqrt{\frac{12h^2 + 3r^2}{20}}$$

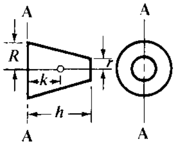


Axis through its center line.

$$k = 0.5477r$$

$$k^2 = 0.3r^2$$

Frustum of Cone:

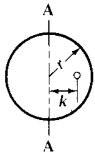


Axis at large end.

$$k = \sqrt{\frac{h^2}{10}\left(\frac{R^2 + 3Rr + 6r^2}{R^2 + Rr + r^2}\right) + \frac{3}{20}\left(\frac{R^5 - r^5}{R^3 - r^3}\right)}$$

Sphere:

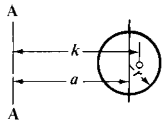
$k = 0.6325r$; $k^2 = \frac{2}{5}r^2$



Axis its diameter.

$$k = \sqrt{a^2 + \tfrac{2}{5}r^2}$$

$$k^2 = a^2 + \tfrac{2}{5}r^2$$

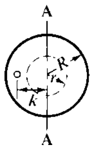


Axis at a distance.

Hollow Sphere and Thin Spherical Shell:

$$k = 0.6325\sqrt{\frac{R^5 - r^5}{R^3 - r^3}}$$

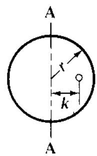
$$k^2 = \frac{2(R^5 - r^5)}{5(R^3 - r^3)}$$



Axis its diameter.

$k = 0.8165r$

$k^2 = \frac{2}{3}r^2$

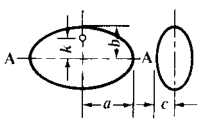


Thin Spherical Shell

Ellipsoid. and Paraboloid:

$$k = 0.447\sqrt{b^2 + c^2}$$

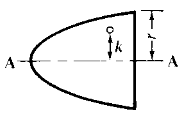
$$k^2 = \tfrac{1}{5}(b^2 + c^2)$$



Axis through center.

$k = 0.5773r$

$k^2 = \frac{1}{3}r^2$



Axis through center.

Center and Radius of Oscillation.—If a body oscillates about a horizontal axis which does not pass through its center of gravity, there will be a point on the line drawn from the center of gravity, perpendicular to the axis, the motion of which will be the same as if the whole mass were concentrated at that point. This point is called the *center of oscillation*. The *radius of oscillation* is the distance between the center of oscillation and the point of suspension. In a straight line, or in a bar of small diameter, suspended at one end and oscillating about it, the center of oscillation is at two-thirds the length of the rod from the end by which it is suspended.

When the vibrations are perpendicular to the plane of the figure, and the figure is suspended by the vertex of an angle or its uppermost point, the radius of oscillation of an isosceles triangle is equal to $3/4$ of the height of the triangle; of a circle, $5/8$ of the diameter; of a parabola, $5/7$ of the height.

If the vibrations are in the plane of the figure, then the radius of oscillation of a circle equals $3/4$ of the diameter; of a rectangle, suspended at the vertex of one angle, $2/3$ of the diagonal.

Center of Percussion.—For a body that moves without rotation, the resultant of all the forces acting on the body passes through the center of gravity. On the other hand, for a body that rotates about some *fixed axis*, the resultant of all the forces acting on it does not pass through the center of gravity of the body but through a point called the *center of percussion*. The center of percussion is useful in determining the position of the resultant in mechanics problems involving angular acceleration of bodies about a fixed axis.

Finding the Center of Percussion when the Radius of Gyration and the Location of the Center of Gravity are Known: The center of percussion lies on a line drawn through the center of rotation and the center of gravity. The distance from the axis of rotation to the center of percussion may be calculated from the following formula

$$q = k_o^2 \div r$$

in which q = distance from the axis of rotation to the center of percussion; k_o = the radius of gyration of the body with respect to the axis of rotation; and r = the distance from the axis of rotation to the center of gravity of the body.

Velocity and Acceleration

Motion is a progressive change of position of a body. Velocity is the rate of motion, that is, the rate of change of position. When the velocity of a body is the same at every moment during which the motion takes place, the latter is called *uniform* motion. When the velocity is variable and constantly increasing, the rate at which it changes is called *acceleration;* that is, acceleration is the rate at which the velocity of a body changes in a unit of time, as the change in feet per second, in one second. When the motion is decreasing instead of increasing, it is called *retarded* motion, and the rate at which the motion is retarded is frequently called the *deceleration*. If the acceleration is uniform, the motion is called *uniformly accelerated* motion. An example of such motion is found in that of falling bodies.

Motion with Constant Velocity.—In the formulas that follow, S = distance moved; V = velocity; t = time of motion, θ = angle of rotation, and ω = angular velocity; the usual units for these quantities are, respectively, feet, feet per second, seconds, radians, and radians per second. Any other consistent set of units may be employed.

Constant Linear Velocity:

$$S = V \times t \qquad V = S \div t \qquad t = S \div V$$

Constant Angular Velocity:

$$\theta = \omega t \qquad \omega = \theta \div t \qquad t = \theta \div \omega$$

Relation between Angular Motion and Linear Motion: The relation between the angular velocity of a rotating body and the linear velocity of a point at a distance r feet from the center of rotation is:

$$V(\text{ft per sec}) = r(\text{ft}) \times \omega(\text{radians per sec})$$

Similarly, the distance moved by the point during rotation through angle θ is:

$$S(\text{ft}) = r(\text{ft}) \times \theta(\text{radians})$$

Linear Motion with Constant Acceleration.—The relations between distance, velocity, and time for linear motion with constant or uniform acceleration are given by the formulas in the accompanying table. In these formulas, the acceleration is assumed to be in the same direction as the initial velocity; hence, if the acceleration in a particular problem should happen to be in a direction opposite that of the initial velocity, then a should be replaced by $-a$. Thus, for example, the formula $V_f = V_o + at$ becomes $V_f = V_o - at$ when a and V_o are opposite in direction.

Linear Motion with Constant Acceleration

To Find	Known	Formula	To Find	Known	Formula
Motion Uniformly Accelerated From Rest					
S	a, t	$S = \frac{1}{2}at^2$	t	S, V_f	$t = 2S \div V_f$
	V_f, t	$S = \frac{1}{2}V_f t$		S, a	$t = \sqrt{2S \div a}$
	V_f, a	$S = V_f^2 \div 2a$		a, V_f	$t = V_f \div a$
V_f	a, t	$V_f = at$	a	S, t	$a = 2S \div t^2$
	S, t	$V_f = 2S \div t$		S, V	$a = V_f^2 \div 2S$
	a, S	$V_f = \sqrt{2aS}$		V_f, t	$a = V_f \div t$
Motion Uniformly Accelerated From Initial Velocity V_o					
S	a, t, V_o	$S = V_o t + \frac{1}{2}at^2$	t	V_o, V_f, a	$t = (V_f - V_o) \div a$
	V_o, V_f, t	$S = (V_f + V_o)t \div 2$		V_o, V_f, S	$t = 2S \div (V_f + V_o)$
	V_o, V_f, a	$S = (V_f^2 - V_o^2) \div 2a$	a	V_o, V_f, S	$a = (V_f^2 - V_o^2) \div 2S$
	V_f, a, t	$S = V_f t - \frac{1}{2}at^2$		V_o, V_f, t	$a = (V_f - V_o) \div t$
V_f	V_o, a, t	$V_f = V_o + at$		V_o, S, t	$a = 2(S - V_o t) \div t^2$
	V_o, S, t	$V_f = (2S \div t) - V_o$		V_f, S, t	$a = 2(V_f t - S) \div t^2$
	V_o, a, S	$V_f = \sqrt{V_o^2 + 2aS}$			
	S, a, t	$V_f = (S \div t) + \frac{1}{2}at$			
V_o	V_f, a, S	$V_o = \sqrt{V_f^2 - 2aS}$			
	V_f, S, t	$V_o = (2S \div t) - V_f$			
	V_f, a, t	$V_o = V_f - at$			
	S, a, t	$V_o = (S \div t) - \frac{1}{2}at$			

Meanings of Symbols

S = distance moved in feet
V_f = final velocity, feet per second
V_o = initial velocity, feet per second
a = acceleration, feet per second per second
t = time of acceleration in seconds

Example: A car is moving at 60 mph when the brakes are suddenly locked and the car begins to skid. If it takes 2 seconds to slow the car to 30 mph, at what rate is it being decelerated, how long is it before the car comes to a halt, and how far will it have traveled?

The initial velocity V_o of the car is 60 mph or 88 ft/sec and the acceleration a due to braking is opposite in direction to V_o, since the car is slowed to 30 mph or 44 ft/sec.

Since V_o, V_f, and t are known, a can be determined from the formula

$$a = (V_f - V_o) \div t = (44 - 88) \div 2$$
$$a = -22 \text{ ft/sec}^2$$

The time required to stop the car can be determined from the formula

$$t = (V_f - V_o) \div a = (0 - 88) \div -22$$
$$t = 4 \text{ seconds}$$

The distance traveled by the car is obtained from the formula

$$S = (V_f + V_o)t \div 2 = (0 + 88)4 \div 2$$
$$= 176 \text{ feet}$$

Rotary Motion with Constant Acceleration.—The relations among angle of rotation, angular velocity, and time for rotation with constant or uniform acceleration are given in the accompanying table.

Rotary Motion with Constant Acceleration

To Find	Known	Formula	To Find	Known	Formula
Motion Uniformly Accelerated From Rest					
θ	α, t	$\theta = \frac{1}{2}\alpha t^2$	t	θ, ω_f	$t = 2\theta \div \omega_f$
	ω_f, t	$\theta = \frac{1}{2}\omega_f t$		θ, α	$t = \sqrt{2\theta \div \alpha}$
	ω_f, α	$\theta = \omega_f^2 \div 2\alpha$		α, ω_f	$t = \omega_f \div \alpha$
ω_f	α, t	$\omega_f = \alpha t$	α	θ, t	$\alpha = 2\theta \div t^2$
	θ, t	$\omega_f = 2\theta \div t$		θ, ω_f	$\alpha = \omega_f^2 \div 2\theta$
	α, θ	$\omega_f = \sqrt{2\alpha\theta}$		ω_f, t	$\alpha = \omega_f \div t$
Motion Uniformly Accelerated From Initial Velocity ω_o					
θ	α, t, ω_o	$\theta = \omega_o t + \frac{1}{2}\alpha t^2$	α	$\omega_o, \omega_f, \theta$	$\alpha = (\omega_f - \omega_o^2) \div 2\theta$
	ω_o, ω_f, t	$\theta = (\omega_f + \omega_o)t \div 2$		ω_o, ω_f, t	$\alpha = (\omega_f - \omega_o) \div t$
	$\omega_o, \omega_f, \alpha$	$\theta = (\omega_f^2 - \omega_o^2) \div 2\alpha$		$\omega_o, \theta\ t$	$\alpha = 2(\theta - \omega_o t) \div t^2$
	ω_f, α, t	$\theta = \omega_f t - \frac{1}{2}\alpha t^2$		$\omega_f, \theta\ t$	$\alpha = 2(\omega_f t - \theta) \div t^2$
ω_f	ω_o, α, t	$\omega_f = \omega_o + \alpha t$	*Meanings of Symbols*		
	ω_o, θ, t	$\omega_f = (2\theta \div t) - \omega_o$			
	ω_o, α, θ	$\omega_f = \sqrt{\omega_o^2 + 2\alpha\theta}$	θ = angle of rotation, radians		
	θ, α, t	$\omega_f = (\theta \div t) + \frac{1}{2}\alpha t$	ω_f = final angular velocity, radians per second		
ω_o	ω_f, α, θ	$\omega_o = \sqrt{\omega_f^2 - 2\alpha\theta}$	ω_o = initial angular velocity, radians per second		
	ω_f, θ, t	$\omega_o = (2\theta \div t) - \omega_f$	α = angular acceleration, radians per second, per second		
	ω_f, α, t	$\omega_o = \omega_f - \alpha t$	t = time in seconds		
	θ, α, t	$\omega_o = (\theta \div t) - \frac{1}{2}\alpha t$			
t	$\omega_o, \omega_f, \alpha$	$t = (\omega_f - \omega_o) \div \alpha$	1 degree = 0.01745 radians		
	$\omega_o, \omega_f, \theta$	$t = 2\theta \div (\omega_f + \omega_o)$	(See conversion table on page 90)		

In these formulas, the acceleration is assumed to be in the same direction as the initial angular velocity; hence, if the acceleration in a particular problem should happen to be in a direction opposite that of the initial angular velocity, then α should be replaced by $-\alpha$. Thus, for example, the formula $\omega_f = \omega_o + \alpha t$ becomes $\omega_f = \omega_o - \alpha t$ when α and ω_o are opposite in direction.

Linear Acceleration of a Point on a Rotating Body: A point on a body rotating about a fixed axis has a linear acceleration a that is the resultant of two component accelerations. The first component is the centripetal or normal acceleration which is directed from the point P toward the axis of rotation; its magnitude is $r\omega^2$ where r is the radius from the axis to the point P and ω is the angular velocity of the body at the time acceleration a is to be determined. The second component of a is the tangential acceleration which is equal to $r\alpha$ where α is the angular acceleration of the body.

The acceleration of point P is the resultant of $r\omega^2$ and $r\alpha$ and is given by the formula

$$a = \sqrt{(r\omega^2)^2 + (r\alpha)^2}$$

When $\alpha = 0$, this formula reduces to: $a = r\omega^2$

Example: A flywheel on a press rotating at 120 rpm is slowed to 102 rpm during a punching operation that requires ¾ second for the punching portion of the cycle. What angular deceleration does the flywheel experience?

From the table on page 187, the angular velocities corresponding to 120 rpm and 102 rpm, respectively, are 12.57 and 10.68 radians per second. Therefore, using the formula

$$\alpha = (\omega_f - \omega_o) \div t$$

$$\alpha = (10.68 - 12.57) \div 3/4 = -1.89 \div 3/4$$

$$\alpha = -2.52 \text{ radians per second per second}$$

which is, from the table on page 187, – 24 rpm per second. The minus sign in the answer indicates that the acceleration α acts to slow the flywheel, that is, the flywheel is being decelerated.

Force, Work, Energy, and Momentum

Accelerations Resulting from Unbalanced Forces.—In the section describing the resolution and composition of forces it was stated that when the resultant of a system of forces is zero, the system is in equilibrium, that is, the body on which the force system acts remains at rest or continues to move with uniform velocity. If, however, the resultant of a system of forces is not zero, the body on which the forces act will be accelerated in the direction of the unbalanced force. To determine the relation between the unbalanced force and the resulting acceleration, Newton's laws of motion must be applied. These laws may be stated as follows:

First Law: Every body continues in a state of rest or in uniform motion in a straight line, until it is compelled by a force to change its state of rest or motion.

Second Law: Change of motion is proportional to the force applied, and takes place along the straight line in which the force acts. The "force applied" represents the resultant of *all* the forces acting on the body. This law is sometimes worded: An unbalanced force acting on a body causes an acceleration of the body in the direction of the force and of magnitude proportional to the force and inversely proportional to the mass of the body. Stated as a formula, $R = Ma$ where R is the resultant of *all* the forces acting on the body, M is the mass of the body (mass = weight W divided by acceleration due to gravity g), and a is the acceleration of the body resulting from application of force R.

Third Law: To every action there is always an equal reaction, or, in other words, if a force acts to change the state of motion of a body, the body offers a resistance equal and directly opposite to the force.

Newton's second law may be used to calculate linear and angular accelerations of a body produced by unbalanced forces and torques acting on the body; however, it is necessary first to use the methods described under "Composition and Resolution of Forces" to determine the magnitude and direction of the resultant of *all* forces acting on the body. Then, for a body moving with pure translation,

$$R = Ma = \frac{W}{g}a$$

where R is the resultant force in pounds acting on a body weighing W pounds; g is the gravitational constant, usually taken as 32.16 ft/sec^2, approximately; and a is the resulting acceleration in ft/sec^2 of the body due to R and in the same direction as R.

Using metric SI units, the formula is $R = Ma$, where R = force in newtons (N), M = mass in kilograms, and a = acceleration in meters/second squared. It should be noted that the weight of a body of mass M kg is Mg newtons, where g is approximately 9.81 m/s^2.

Free Body Diagram: In order to correctly determine the effect of forces on the motion of a body it is necessary to resort to what is known as a *free body diagram*. This diagram shows 1) the body removed or isolated from contact with all other bodies that exert force on the body and; and 2) *all* the forces acting on the body.

Thus, for example, in Fig. a the block being pulled up the plane is acted upon by certain forces; the free body diagram of this block is shown at Fig. b. Note that all forces acting on the block are indicated. These forces include: 1) the force of gravity (weight); 2) the pull of the cable, P; 3) the normal component, $W \cos \phi$, of the force exerted on the block by the plane; and 4) the friction force, $\mu W \cos \phi$, of the plane on the block.

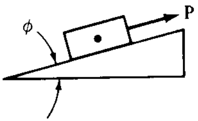


Fig. a.

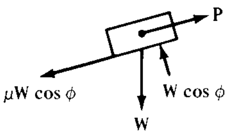


Fig. b.

In preparing a free body diagram, it is important to realize that only those forces exerted *on* the body being considered are shown; forces exerted by the body on other bodies are disregarded. This feature makes the free body diagram an invaluable aid in the solution of problems in mechanics.

Example: A 100-pound body is being hoisted by a winch, the tension in the hoisting cable being kept constant at 110 pounds. At what rate is the body accelerated?

Two forces are acting on the body, its weight, 100 pounds downward, and the pull of the cable, 110 pounds upward. The resultant force R, from a free body diagram, is therefore 110 – 100. Thus, applying Newton's second law,

$$110 - 100 = \frac{100}{32.16}a$$

$$a = \frac{32.16 \times 10}{100} = 3.216 \text{ ft/sec}^2 \text{ upward}$$

It should be noted that since in this problem the resultant force *R* was positive (110 – 100 = + 10), the acceleration *a* is also positive, that is, *a* is in the same direction as *R*, which is in accord with Newton's second law.

***Example using SI metric units:* A body of mass 50 kilograms is being hoistedby a winch, and the tension in the cable is 600 newtons. What is the acceleration? The weight of the 50 kg body is 50*g* newtons, where *g* = approximately 9.81 m/s² (see *Note* on page 163). Applying the formula *R* = *Ma*, the calculation is: (600 – 50*g*) = 50*a*. Thus,**

$$a = \frac{600 - 50g}{50} = \frac{600 - (50 \times 9.81)}{50} = 2.19 \text{ m/s}^2$$

Formulas Relating Torque and Angular Acceleration: For a body rotating about a fixed axis the relation between the unbalanced torque acting to produce rotation and the resulting angular acceleration may be determined from any one of the following formulas, each based on Newton's second law:

$$T_o = J_M \alpha$$

$$T_o = M k_o^2 \alpha$$

$$T_o = \frac{W k_o^2 \alpha}{g} = \frac{W k_o^2 \alpha}{32.16}$$

where T_o is the unbalanced torque in pounds-feet; J_M in ft-lbs-sec² is the moment of inertia of the body about the axis of rotation; k_o in feet is the radius of gyration of the body with respect to the axis of rotation, and α in radians per second, per second is the angular acceleration of the body.

Example: A flywheel has a diameter of 3 feet and weighs 1000 pounds. What torque must be applied, neglecting bearing friction, to accelerate the flywheel at the rate of 100 revolutions per minute, per second?

From page 144 the moment of inertia of a solid cylinder with respect to a gravity axis at right angles to the circular cross-section is given as ½ Mr^2. From page 187, 100 rpm = 10.47 radians per second, hence an acceleration of 100 rpm per second = 10.47 radians per second, per second. Therefore, using the first of the preceding formulas,

$$T_o = J_M \alpha = \left(\frac{1}{2}\right)\frac{1000}{32.16}\left(\frac{3}{2}\right)^2 \times 10.47$$

$$= 366 \text{ ft-lbs}$$

Using metric SI units, the formulas are: $T_o = J_M\alpha = Mk_o^2\alpha$, where T_o = torque in newton-meters; J_M = the moment of inertia in kg · m², and α = the angular acceleration in radians per second squared.

***Example:* A flywheel has a diameter of 1.5 m, and a mass of 800 kg. What torque is needed to produce an angular acceleration of 100 revolutions per minute, per second? As in the preceding example, α = 10.47 rad/s². Thus:**

$$J_M = \tfrac{1}{2}Mr^2 = \tfrac{1}{2} \times 800 \times 0.75^2 = 225 \text{ kg} \cdot \text{m}^2$$

Therefore: $T_o = J_M\alpha = 225 \times 10.47 = 2356$ N · m.

Energy.—A body is said to possess energy when it is capable of doing work or overcoming resistance. The energy may be either mechanical or non-mechanical, the latter including chemical, electrical, thermal, and atomic energy.

Mechanical energy includes *kinetic energy* (energy possessed by a body because of its motion) and *potential energy* (energy possessed by a body because of its position in a field of force and/or its elastic deformation).

Kinetic Energy: The motion of a body may be one of pure translation, pure rotation, or a combination of rotation and translation. By translation is meant motion in which every line in the body remains parallel to its original position throughout the motion, that is, no rotation is associated with the motion of the body.

The kinetic energy of a translating body is given by the formula

$$\text{Kinetic Energy in ft lbs due to translation} = E_{KT} = \tfrac{1}{2}MV^2 = \frac{WV^2}{2g} \quad \text{(a)}$$

where M = mass of body (= $W \div g$); V = velocity of the center of gravity of the body in feet per second; W = weight of body in pounds; and g = acceleration due to gravity = 32.16 feet per second, per second.

The kinetic energy of a body rotating about a fixed axis O is expressed by the formula:

$$\text{Kinetic Energy in ft lbs due to rotation} = E_{KR} = \tfrac{1}{2}J_{MO}\omega^2 \quad \text{(b)}$$

where J_{MO} is the moment of inertia of the body about the fixed axis O in pounds-feet-seconds^2, and ω = angular velocity in radians per second.

For a body that is moving with both translation and rotation, the total kinetic energy is given by the following formula as the sum of the kinetic energy due to translation of the center of gravity and the kinetic energy due to rotation about the center of gravity:

$$\begin{aligned}\text{Total Kinetic Energy in ft lbs} &= E_T = \tfrac{1}{2}MV^2 + \tfrac{1}{2}J_{MG}\omega^2 \\ &= \frac{WV^2}{2g} + \tfrac{1}{2}J_{MG}\omega^2 \\ &= \frac{WV^2}{2g} + \frac{1}{2}\frac{Wk^2\omega^2}{g} \quad \text{(c)} \\ &= \frac{W}{2g}(V^2 + k^2\omega^2)\end{aligned}$$

where J_{MG} is the moment of inertia of the body about its gravity axis in pounds-feet-seconds^2, k is the radius of gyration in feet with respect to an axis through the center of gravity, and the other quantities are as previously defined.

In the metric SI system, energy is expressed as the joule (J). One joule = 1 newton-meter. The kinetic energy of a translating body is given by the formula $E_{KT} = \tfrac{1}{2}MV^2$, where M = mass in kilograms, and V = velocity in meters per second. Kinetic energy due to rotation is expressed by the formula $E_{KR} = \tfrac{1}{2}J_{MO}\omega^2$, where J_{MO} = moment of inertia in kg · m^2, and ω = the angular velocity in radians per second. Total kinetic energy $ET = \tfrac{1}{2}MV^2 + \tfrac{1}{2}J_{MO}\omega^2$ joules = $\tfrac{1}{2}M(V^2 + k^2\omega^2)$ joules, where k = radius of gyration in meters.

Potential Energy: The most common example of a body having potential energy because of its position in a field of force is that of a body elevated to some height above the earth. Here the field of force is the gravitational field of the earth and the potential energy E_{PF} of a body weighing W pounds elevated to some height S in feet above the surface of the earth is WS foot-pounds. If the body is permitted to drop from this height its potential energy E_{PF} will be converted to kinetic energy. Thus, after falling through height S the kinetic energy of the body will be WS ft-lbs.

In metric SI units, the potential energy E_{PF} of a body of mass M kilograms elevated to a height of S meters, is MgS joules. After it has fallen a distance S, the kinetic energy gained will thus be MgS joules.

Another type of potential energy is elastic potential energy, such as possessed by a spring that has been compressed or extended. The amount of work in ft lbs done in compressing the spring S feet is equal to $KS^2/2$, where K is the spring constant in pounds per foot. Thus, when the spring is released to act against some resistance, it can perform $KS^2/2$ ft-lbs of work which is the amount of elastic potential energy E_{PE} stored in the spring.

Using metric SI units, the amount of work done in compressing the spring a distance S meters is $KS^2/2$ joules, where K is the spring constant in newtons per meter.

Work Performed by Forces and Couples.—The work U done by a force F in moving an object along some path is the product of the distance S the body is moved and the component $F \cos \alpha$ of the force F in the direction of S.

$$U = FS\cos\alpha$$

where U = work in ft-lbs; S = distance moved in feet; F = force in lbs; and α = angle between line of action of force and the path of S.

If the force is in the same direction as the motion, then $\cos\alpha = \cos 0 = 1$ and this formula reduces to:

$$U = FS$$

Similarly, the work done by a couple T turning an object through an angle θ is:

$$U = T\theta$$

where T = torque of couple in pounds-feet and θ = the angular rotation in radians.

The above formulas can be used with metric SI units: U is in joules; S is in meters; F is in newtons, and T is in newton-meters.

Relation between Work and Energy.—Theoretically, when work is performed on a body and there are no energy losses (such as due to friction, air resistance, etc.), the energy acquired by the body is equal to the work performed on the body; this energy may be either potential, kinetic, or a combination of both.

In actual situations, however, there may be energy losses that must be taken into account. Thus, the relation between work done on a body, energy losses, and the energy acquired by the body can be stated as:

$$\text{Work Performed} - \text{Losses} = \text{Energy Acquired}$$

$$U - \text{Losses} = E_T$$

Example 1: A 12-inch cube of steel weighing 490 pounds is being moved on a horizontal conveyor belt at a speed of 6 miles per hour (8.8 feet per second). What is the kinetic energy of the cube?

Since the block is not rotating, Formula (a) for the kinetic energy of a body moving with pure translation applies:

$$\text{Kinetic Energy} = \frac{WV^2}{2g}$$

$$= \frac{490 \times (8.8)^2}{2 \times 32.16} = 590 \text{ ft-lbs}$$

A similar example using metric SI units is as follows: If a cube of mass 200 kg is being moved on a conveyor belt at a speed of 3 meters per second, what is the kinetic energy of the cube? It is:

$$\textbf{Kinetic Energy} = \tfrac{1}{2}MV^2 = \tfrac{1}{2} \times 200 \times 3^2 = 900 \textbf{ joules}$$

Example 2: If the conveyor in Example 1 is brought to an abrupt stop, how long would it take for the steel block to come to a stop and how far along the belt would it slide before stopping if the coefficient of friction μ between the block and the conveyor belt is 0.2 and the block slides without tipping over?

The only force acting to slow the motion of the block is the friction force between the block and the belt. This force F is equal to the weight of the block, W, multiplied by the coefficient of friction; $F = \mu W = 0.2 \times 490 = 98$ lbs.

The time required to bring the block to a stop can be determined from the impulse-momentum Formula (c) on page 160.

$$R \times t = \frac{W}{g}(V_f - V_o)$$

$$(-98)t = \frac{490}{32.16} \times (0 - 8.8)$$

$$t = \frac{490 \times 8.8}{98 \times 32.16} = 1.37 \text{ seconds}$$

The distance the block slides before stopping can be determined by equating the kinetic energy of the block and the work done by friction in stopping it:

$$\text{Kinetic energy of block}(WV^2/2g) = \text{Work done by friction}(F \times S)$$

$$590 = 98 \times S$$

$$S = \frac{590}{98} = 6.0 \text{ feet}$$

If metric SI units are used, the calculation is as follows (for the cube of 200 kg mass): The friction force = μ multiplied by the weight Mg where g = approximately 9.81 m/s^2. Thus, $\mu Mg = 0.2 \times 200g = 392.4$ newtons. The time t required to bring the block to a stop is $(-392.4)t = 200(0 - 3)$. Therefore,

$$t = \frac{200 \times 3}{392.4} = 1.53 \text{ seconds}$$

The kinetic energy of the block is equal to the work done by friction, that is $392.4 \times S = 900$ joules. Thus, the distance S which the block moves before stopping is

$$S = \frac{900}{392.4} = 2.29 \text{ meters}$$

Force of a Blow.—A body that weighs W pounds and falls S feet from an initial position of rest is capable of doing WS foot-pounds of work. The work performed during its fall may be, for example, that necessary to drive a pile a distance d into the ground. Neglecting losses in the form of dissipated heat and strain energy, the work done in driving the pile is equal to the product of the impact force acting on the pile and the distance d which the pile is driven. Since the impact force is not accurately known, an average value, called the "average force of the blow," may be assumed. Equating the work done on the pile and the work done by the falling body, which in this case is a pile driver:

$$\text{Average force of blow} \times d = WS$$

or,

$$\text{Average force of blow} = \frac{WS}{d}$$

where, S = total height in feet through which the driver falls, including the distance d that the pile is driven

W = weight of driver in pounds

d = distance in feet which pile is driven

When using metric SI units, it should be noted that a body of mass *M* kilograms has a weight of *Mg* newtons, where *g* = approximately 9.81 m/s². If the body falls a distance *S* meters, it can do work equal to *MgS* joules. The average force of the blow is *MgS*/*d* newtons, where *d* is the distance in meters that the pile is driven.

Example: A pile driver weighing 200 pounds strikes the top of the pile after having fallen from a height of 20 feet. It forces the pile into the ground a distance of ½ foot. Before the ram is brought to rest, it will $200 \times (20 + \frac{1}{2}) = 4100$ foot-pounds of work, and as this energy is expended in a distance of one-half foot, the average force of the blow equals $4100 \div \frac{1}{2} =$ 8200 pounds.

A similar example using metric SI units is as follows: A pile driver of mass 100 kilograms falls 10 meters and moves the pile a distance of 0.3 meters. The work done = 100*g*(10 + 0.3) joules, and it is expended in 0.3 meters. Thus, the average force is

$$\frac{100g \times 10.3}{0.3} = 33680 \text{ newtons or } 33.68 \text{ kN}$$

Impulse and Momentum.—The *linear momentum* of a body is defined as the product of the mass *M* of the body and the velocity *V* of the center of gravity of the body:

$$\text{Linear momentum} = MV \text{ or since } M = W \div g$$

$$\text{Linear momentum} = \frac{WV}{g} \qquad \text{(a)}$$

It should be noted that linear momentum is a vector quantity, the momentum being in the same direction as *V*.

Linear impulse: is defined as the product of the resultant *R* of *all* the forces acting on a body and the time *t* that the resultant acts:

$$\text{Linear Impulse} = Rt \qquad \text{(b)}$$

The change in the linear momentum of a body is numerically equal to the linear impulse that causes the change in momentum:

$$\text{Linear Impulse} = \text{change in Linear Momentum}$$

$$Rt = \frac{W}{g}V_f - \frac{W}{g}V_o = \frac{W}{g}(V_f - V_o) \qquad \text{(c)}$$

where V_f, the final velocity of the body after time *t*, and V_o, the initial velocity of the body, are both in the same direction as the applied force *R*. If V_o, and V_f are in opposite directions, then the minus sign in the formula becomes a plus sign.

In metric SI units, the formulas are: Linear Momentum = *MV* kg · m/s, where *M* = mass in kg, and *V* = velocity in meters per second; and Linear Impulse = *Rt* newton-seconds, where *R* = force in newtons, and *t* = time in seconds. In Formula (c) above, *W*/*g* is replaced by *M* when SI units are used.

Example: A 1000-pound block is pulled up a 2-degree incline by a cable exerting a constant force *F* of 600 pounds. If the coefficient of friction μ between the block and the plane is 0.5, how fast will the block be moving up the plane 10 seconds after the pull is applied?

The resultant force *R* causing the body to be accelerated up the plane is the difference between *F*, the force acting up the plane, and *P*, the force acting to resist motion up the plane. This latter force for a body on a plane is given by the formula at the top of page 130 as $P = W(\mu \cos \alpha + \sin \alpha)$ where α is the angle of the incline.

Thus, $R = F - P = F - W(\mu \cos \alpha + \sin \alpha)$

$$= 600 - 1000(0.5 \cos 2^\circ + \sin 2^\circ)$$

$$= 600 - 1000(0.5 \times 0.99939 + 0.03490)$$

$$= 600 - 535$$

$$R = 65 \text{ pounds.}$$

Formula (c) can now be applied to determine the speed at which the body will be moving up the plane after 10 seconds.

$$Rt = \frac{W}{g}V_f - \frac{W}{g}V_o$$

$$65 \times 10 = \frac{1000}{32.2}V_f - \frac{1000}{32.2} \times 0$$

$$V_f = \frac{65 \times 10 \times 32.2}{1000} = 20.9 \text{ ft per sec}$$

$$= 14.3 \text{ miles per hour}$$

A similar example using metric SI units is as follows: A 500 kg block is pulled up a 2 degree incline by a constant force *F* of 4 kN. The coefficient of friction μ between the block and the plane is 0.5. How fast will the block be moving 10 seconds after the pull is applied?

The resultant force *R* is:

$$\mathbf{R = F - Mg(\mu\cos\alpha + \sin\alpha)}$$

$$\mathbf{= 4000 - 500 \times 9.81(0.5 \times 0.99939 + 0.03490)}$$

$$\mathbf{= 1378 N \text{ or } 1.378 \text{ kN}}$$

Formula (c) can now be applied to determine the speed at which the body will be moving up the plane after 10 seconds. Replacing *W/g* by *M* in the formula, the calculation is:

$$\mathbf{Rt = MV_f - MV_o}$$

$$\mathbf{1378 \times 10 = 500(V_f - 0)}$$

$$\mathbf{V_f = \frac{1378 \times 10}{500} = 27.6 \text{ m/s}}$$

Angular Impulse and Momentum: In a manner similar to that for linear impulse and moment, the formulas for angular impulse and momentum for a body rotating about a fixed axis are:

$$\text{Angular momentum} = J_M\omega \qquad \text{(a)}$$

$$\text{Angular impulse} = T_o t \qquad \text{(b)}$$

where J_M is the moment of inertia of the body about the axis of rotation in pounds-feet-seconds^2, ω is the angular velocity in radians per second, T_o, is the torque in pounds-feet about the axis of rotation, and *t* is the time in seconds that T_o, acts.

The change in angular momentum of a body is numerically equal to the angular impulse that causes the change in angular momentum:

$$\text{Angular Impulse} = \text{Change in Angular Momentum}$$

$$T_o t = J_M\omega_f - J_M\omega_o = J_M(\omega_f - \omega_o) \qquad \text{(c)}$$

where ω_f and ω_o are the final and initial angular velocities, respectively.

Example: A flywheel having a moment of inertia of 25 lbs-ft-sec^2 is revolving with an angular velocity of 10 radians per second when a constant torque of 20 lbs-ft is applied to reverse its direction of rotation. For what length of time must this constant torque act to stop the flywheel and bring it up to a reverse speed of 5 radians per second?

Applying Formula (c),

$$T_o t = J_M(\omega_f - \omega_o)$$
$$20t = 25(10 - [-5]) = 250 + 125$$
$$t = 375 \div 20 = 18.8 \text{ seconds}$$

A similar example using metric SI units is as follows: A flywheel with a moment of inertia of 20 kilogram-meters^2 is revolving with an angular velocity of 10 radians per second when a constant torque of 30 newton-meters is applied to reverse its direction of rotation. For what length of time must this constant torque act to stop the flywheel and bring it up to a reverse speed of 5 radians per second? Applying Formula (c), the calculation is:

$$\boldsymbol{T_o t = J_M(\omega_f - \omega_o),}$$
$$\boldsymbol{30t = 20(10 - [-5]).}$$
$$\textbf{Thus, } \boldsymbol{t = \frac{20 \times 15}{30} = 10} \textbf{ seconds}$$

Formulas for Work and Power.—The formulas in the accompanying table may be used to determine work and power in terms of the applied force and the velocity at the point of application of the force.

Formulas for Work and Power

To Find	Known	Formula	To Find	Known	Formula
S	P, t, F	$S = P \times t \div F$	P	F, V	$P = F \times V$
	K, F	$S = K \div F$		F, S, t	$P = F \times S \div t$
	t, F, hp	$S = 550 \times t \times hp \div F$		K, t	$P = K \div t$
V	P, F	$V = P \div F$		hp	$P = 550 \times hp$
	K, F, t	$V = K \div (F \times t)$	K	F, S	$K = F \times S$
	F, hp	$V = 550 \times hp \div F$		P, t	$K = P \times t$
t	F, S, P	$t = F \times S \div P$		F, V, t	$K = F \times V \times t$
	K, F, V	$t = K \div (F \times V)$		t, hp	$K = 550 \times t \times hp$
	F, S, hp	$t = F \times S \div (550 \times hp)$	hp	F, S, t	$hp = F \times S \div (550 \times t)$
F	P, V	$F = P \div V$		P	$hp = P \div 550$
	K, S	$F = K \div S$		F, V	$hp = F \times V \div 550$
	K, V, t	$F = K \div (V \times t)$		K, t	$hp = K \div (550 \times t)$
	V, hp	$F = 550 \times hp \div V$			

Meanings of Symbols: S = distance in feet; V = constant or average velocity in feet per second; t = time in seconds; F = constant or average force in pounds; P = power in foot-pounds per second; K = work in foot-pounds; and hp = horsepower.

***Note:* The metric SI unit of work is the joule (one joule = 1 newton-meter), and the unit of power is the watt (one watt = 1 joule per second = 1 N · m/s). The term horsepower is not used. Thus, those formulas above that involve horsepower and the factor 550 are not applicable when working in SI units. The remaining formulas can be used, and the units are: *S* = distance**

in meters; V = constant or average velocity in meters per second; t = time in seconds; F = force in newtons; P = power in watts; K = work in joules.

Example: A casting weighing 300 pounds is to be lifted by means of an overhead crane. The casting is lifted 10 feet in 12 seconds. What is the horsepower developed? Here F = 300; S = 10; t = 12.

$$\text{hp} = \frac{F \times S}{550t} = \frac{300 \times 10}{550 \times 12} = 0.45$$

A similar example using metric SI units is as follows: A casting of mass 150 kg is lifted 4 meters in 15 seconds by means of a crane. What is the power? Here $F = 150g$ N, S = 4 m, and t = 15 s. Thus:

$$\text{Power} = \frac{FS}{t} = \frac{150g \times 4}{15} = \frac{150 \times 9.81 \times 4}{15}$$
$$= 392 \text{ watts or } 0.392 \text{ kW}$$

Centrifugal Force

Centrifugal Force.—When a body rotates about any axis other than one at its center of mass, it exerts an outward radial force called centrifugal force upon the axis or any arm or cord from the axis that restrains it from moving in a straight (tangential) line. In the following formulas:

F = centrifugal force in pounds
W = weight of revolving body in pounds
v = velocity at radius R on body in feet per second
n = number of revolutions per minute
g = acceleration due to gravity = 32.16 feet per second per second
R = perpendicular distance in feet from axis of rotation to center of mass, or for practical use, to center of gravity of revolving body

Note: If a body rotates about its own center of mass, R equals zero and v equals zero. This means that the *resultant* of the centrifugal forces of all the elements of the body is equal to zero or, in other words, no centrifugal force is exerted on the axis of rotation. The centrifugal force of any part or element of such a body is found by the equations given below, where R is the radius to the center of gravity of the part or element. In a flywheel rim, R is the mean radius of the rim because it is the radius to the center of gravity of a thin radial section.

$$F = \frac{Wv^2}{gR} = \frac{Wv^2}{32.16R} = \frac{4WR\pi^2 n^2}{60 \times 60g} = \frac{WRn^2}{2933} = 0.000341\,WRn^2$$

$$W = \frac{FRg}{v^2} = \frac{2933F}{Rn^2} \qquad v = \sqrt{\frac{FRg}{W}}$$

$$R = \frac{Wv^2}{Fg} = \frac{2933F}{Wn^2} \qquad n = \sqrt{\frac{2933F}{WR}}$$

(If n is the number of revolutions per second instead of per minute, then $F = 1227WRn^2$.)

If metric SI units are used in the foregoing formulas, W/g is replaced by M, which is the mass in kilograms; F = centrifugal force in newtons; v = velocity in meters per second; n = number of revolutions per minute; and R = the radius in meters. Thus:

$$F = Mv^2/R = \frac{Mn^2(2\pi R^2)}{60^2 R} = 0.01097\ MRn^2$$

If the rate of rotation is expressed as n_1 = revolutions per second, then F = 39.48 MRn_1^2; if it is expressed as ω radians per second, then $F = MR\omega^2$.

Calculating Centrifugal Force.—In the ordinary formula for centrifugal force, F = 0.000341 WRn^2; the mean radius R of the flywheel or pulley rim is given in feet. For small dimensions, it is more convenient to have the formula in the form:

$$F = 0.2842 \times 10^{-4} Wrn^2$$

in which F = centrifugal force, in pounds; W = weight of rim, in pounds; r = mean radius of rim, in inches; n = number of revolutions per minute.

In this formula let $C = 0.000028416n^2$. This, then, is the centrifugal force of one pound, one inch from the axis. The formula can now be written in the form,

$$F = WrC$$

C is calculated for various values of the revolutions per minute n, and the calculated values of C are given in Table 1. To find the centrifugal force in any given case, simply find the value of C in the table and multiply it by the product of W and r, the four multiplications in the original formula given thus having been reduced to two.

Example: A cast-iron flywheel with a mean rim radius of 9 inches, is rotated at a speed of 800 revolutions per minute. If the weight of the rim is 20 pounds, what is the centrifugal force?

From Table 1, for n = 800 revolutions per minute, the value of C is 18.1862.

Thus,

$$\begin{aligned} F &= WrC \\ &= 20 \times 9 \times 18.1862 \\ &= 3273.52 \text{ pounds} \end{aligned}$$

Using metric SI units, 0.01097n^2 is the centrifugal force acting on a body of 1 kilogram mass rotating at n revolutions per minute at a distance of 1 meter from the axis. If this value is designated C_1, then the centrifugal force of mass M kilograms rotating at this speed at a distance from the axis of R meters, is C_1MR newtons. To simplify calculations, values for C_1 are given in Table 2. If it is required to work in terms of millimeters, the force is 0.001 C_1MR_1 newtons, where R_1 is the radius in millimeters.

***Example:* A steel pulley with a mean rim radius of 120 millimeters is rotated at a speed of 1100 revolutions per minute. If the mass of the rim is 5 kilograms, what is the centrifugal force?**

From Table 2, for n = 1100 revolutions per minute, the value of C_1 is 13,269.1.

Thus,

$$\begin{aligned} F &= 0.001\ C_1MR_1 \\ &= 0.001 \times 13{,}269.1 \times 5 \times 120 \\ &= 7961.50 \text{ newtons} \end{aligned}$$

Table 1. Factors *C* for Calculating Centrifugal Force (English units)

n	*C*	*n*	*C*	*n*	*C*	*n*	*C*
50	0.07104	100	0.28416	470	6.2770	5200	768.369
51	0.07391	101	0.28987	480	6.5470	5300	798.205
52	0.07684	102	0.29564	490	6.8227	5400	828.611
53	0.07982	103	0.30147	500	7.1040	5500	859.584
54	0.08286	104	0.30735	600	10.2298	5600	891.126
55	0.08596	105	0.31328	700	13.9238	5700	923.236
56	0.08911	106	0.31928	800	18.1862	5800	955.914
57	0.09232	107	0.32533	900	23.0170	5900	989.161
58	0.09559	108	0.33144	1000	28.4160	6000	1022.980
59	0.09892	109	0.33761	1100	34.3834	6100	1057.360
60	0.10230	110	0.34383	1200	40.9190	6200	1092.310
61	0.10573	115	0.37580	1300	48.0230	6300	1127.830
62	0.10923	120	0.40921	1400	55.6954	6400	1163.920
63	0.11278	125	0.44400	1500	63.9360	6500	1200.580
64	0.11639	130	0.48023	1600	72.7450	6600	1237.800
65	0.12006	135	0.51788	1700	82.1222	6700	1275.590
66	0.12378	140	0.55695	1800	92.0678	6800	1313.960
67	0.12756	145	0.59744	1900	102.5820	6900	1352.890
68	0.13140	150	0.63936	2000	113.6640	7000	1392.380
69	0.13529	160	0.72745	2100	125.3150	7100	1432.450
70	0.13924	170	0.82122	2200	137.5330	7200	1473.090
71	0.14325	180	0.92067	2300	150.3210	7300	1514.290
72	0.14731	190	1.02590	2400	163.6760	7400	1556.060
73	0.15143	200	1.1367	2500	177.6000	7500	1598.400
74	0.15561	210	1.2531	2600	192.0920	7600	1641.310
75	0.15984	220	1.3753	2700	207.1530	7700	1684.780
76	0.16413	230	1.5032	2800	222.7810	7800	1728.830
77	0.16848	240	1.6358	2900	238.9790	7900	1773.440
78	0.17288	250	1.7760	3000	255.7400	8000	1818.620
79	0.17734	260	1.9209	3100	273.0780	8100	1864.370
80	0.18186	270	2.0715	3200	290.9800	8200	1910.690
81	0.18644	280	2.2278	3300	309.4500	8300	1957.580
82	0.19107	290	2.3898	3400	328.4890	8400	2005.030
83	0.19576	300	2.5574	3500	348.0960	8500	2053.060
84	0.20050	310	2.7308	3600	368.2710	8600	2101.650
85	0.20530	320	2.9098	3700	389.0150	8700	2150.810
86	0.21016	330	3.0945	3800	410.3270	8800	2200.540
87	0.21508	340	3.2849	3900	432.2070	8900	2250.830
88	0.22005	350	3.4809	4000	454.6560	9000	2301.700
89	0.22508	360	3.6823	4100	477.6730	9100	2353.130
90	0.23017	370	3.8901	4200	501.2580	9200	2405.130
91	0.23531	380	4.1032	4300	525.4120	9300	2457.700
92	0.24051	390	4.3220	4400	550.1340	9400	2510.840
93	0.24577	400	4.5466	4500	575.4240	9500	2564.540
94	0.25108	410	4.7767	4600	601.2830	9600	2618.820
95	0.25645	420	5.0126	4700	627.7090	9700	2673.660
96	0.26188	430	5.2541	4800	654.7050	9800	2729.070
97	0.26737	440	5.5013	4900	682.2680	9900	2785.050
98	0.27291	450	5.7542	5000	710.4000	10000	2841.600
99	0.27851	460	6.0128	5100	739.1000	…	…

Table 2. Factors C_1 for Calculating Centrifugal Force (Metric SI units)

n	C_1	n	C_1	n	C_1	n	C_1
50	27.4156	100	109.662	470	2,422.44	5200	296,527
51	28.5232	101	111.867	480	2,526.62	5300	308,041
52	29.6527	102	114.093	490	2,632.99	5400	319,775
53	30.8041	103	116.341	500	2,741.56	5500	331,728
54	31.9775	104	118.611	600	3,947.84	5600	343,901
55	33.1728	105	120.903	700	5,373.45	5700	356,293
56	34.3901	106	123.217	800	7,018.39	5800	368,904
57	35.6293	107	125.552	900	8,882.64	5900	381,734
58	36.8904	108	127.910	1000	10,966.2	6000	394,784
59	38.1734	109	130.290	1100	13,269.1	6100	408,053
60	39.4784	110	132.691	1200	15,791.4	6200	421,542
61	40.8053	115	145.028	1300	18,532.9	6300	435,250
62	42.1542	120	157.914	1400	21,493.8	6400	449,177
63	43.5250	125	171.347	1500	24,674.0	6500	463,323
64	44.9177	130	185.329	1600	28,073.5	6600	477,689
65	46.3323	135	199.860	1700	31,692.4	6700	492,274
66	47.7689	140	214.938	1800	35,530.6	6800	507,078
67	49.2274	145	230.565	1900	39,588.1	6900	522,102
68	50.7078	150	246.740	2000	43,864.9	7000	537,345
69	52.2102	160	280.735	2100	48,361.1	7100	552,808
70	53.7345	170	316.924	2200	53,076.5	7200	568,489
71	55.2808	180	355.306	2300	58,011.3	7300	584,390
72	56.8489	190	395.881	2400	63,165.5	7400	600,511
73	58.4390	200	438.649	2500	68,538.9	7500	616,850
74	60.0511	210	483.611	2600	74,131.7	7600	633,409
75	61.6850	220	530.765	2700	79,943.8	7700	650,188
76	63.3409	230	580.113	2800	85,975.2	7800	667,185
77	65.0188	240	631.655	2900	92,226.0	7900	684,402
78	66.7185	250	685.389	3000	98,696.0	8000	701,839
79	68.4402	260	741.317	3100	105,385	8100	719,494
80	70.1839	270	799.438	3200	112,294	8200	737,369
81	71.9494	280	859.752	3300	119,422	8300	755,463
82	73.7369	290	922.260	3400	126,770	8400	773,777
83	75.5463	300	986.960	3500	134,336	8500	792,310
84	77.3777	310	1,053.85	3600	142,122	8600	811,062
85	79.2310	320	1,122.94	3700	150,128	8700	830,034
86	81.1062	330	1,194.22	3800	158,352	8800	849,225
87	83.0034	340	1,267.70	3900	166,796	8900	868,635
88	84.9225	350	1,343.36	4000	175,460	9000	888,264
89	86.8635	360	1,421.22	4100	184,342	9100	908,113
90	88.8264	370	1,501.28	4200	193,444	9200	928,182
91	90.8113	380	1,583.52	4300	202,766	9300	948,469
92	92.8182	390	1,667.96	4400	212,306	9400	968,976
93	94.8469	400	1,754.60	4500	222,066	9500	989,702
94	96.8976	410	1,843.42	4600	232,045	9600	1,010,650
95	98.9702	420	1,934.44	4700	242,244	9700	1,031,810
96	101.065	430	2,027.66	4800	252,662	9800	1,053,200
97	103.181	440	2,123.06	4900	263,299	9900	1,074,800
98	105.320	450	2,220.66	5000	274,156	10000	1,096,620
99	107.480	460	2,320.45	5100	285,232	…	…

Balancing Rotating Parts

Static Balancing.—There are several methods of testing the standing or static balance of a rotating part. A simple method that is sometimes used for flywheels, etc., is illustrated by the diagram, Fig. 1. An accurate shaft is inserted through the bore of the finished wheel, which is then mounted on carefully leveled "parallels" A. If the wheel is in an unbalanced state, it will turn until the heavy side is downward. When it will stand in any position as the result of counterbalancing and reducing the heavy portions, it is said to be in standing or static balance. Another test which is used for disk-shaped parts is shown in Fig. 2. The disk D is mounted on a vertical arbor attached to an adjustable cross-slide B. The latter is carried by a table C, which is supported by a knife-edged bearing. A pendulum having an adjustable screw-weight W at the lower end is suspended from cross-slide B. To test the static balance of disk D, slide B is adjusted until pointer E of the pendulum coincides with the center of a stationary scale F. Disk D is then turned halfway around without moving the slide, and if the indicator remains stationary, it shows that the disk is in balance for this particular position. The test is then repeated for ten or twelve other positions, and the heavy sides are reduced, usually by drilling out the required amount of metal. Several other devices for testing static balance are designed on this same principle.

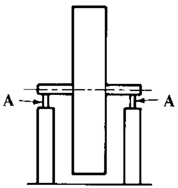


Fig. 1.

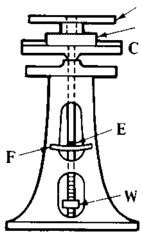


Fig. 2.

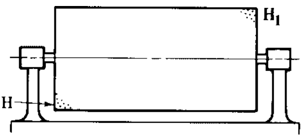


Fig. 3.

Running or Dynamic Balance.—A cylindrical body may be in perfect static balance and not be in a balanced state when rotating at high speed. If the part is in the form of a thin disk, static balancing, if carefully done, may be accurate enough for high speeds, but if the rotating part is long in proportion to its diameter, and the unbalanced portions are at opposite ends or in different planes, the balancing must be done so as to counteract the centrifugal force of these heavy parts when they are rotating rapidly. This process is known as a running balance or dynamic balancing. To illustrate, if a heavy section is located at H (Fig. 3), and another correspondingly heavy section at H_1, one may exactly counterbalance the other when the cylinder is stationary, and this static balance may be sufficient for a part rigidly mounted and rotating at a comparatively slow speed; but when the speed is very high, as in turbine rotors, etc., the heavy masses H and H_1, being in different planes, are in an unbalanced state owing to the effect of centrifugal force, which results in excessive strains and injurious vibrations. Theoretically, to obtain a perfect running balance, the exact positions of the heavy sections should be located and the balancing effected either by reducing their weight or by adding counterweights opposite each section and in the same plane at the proper radius; but if the rotating part is rigidly mounted on a stiff shaft, a running balance that is sufficiently accurate for practical purposes can be obtained by means of comparatively few counterbalancing weights located with reference to the unbalanced parts.

Balancing Calculations.—As indicated previously, centrifugal forces caused by an unbalanced mass or masses in a rotating machine member cause additional loads on the bearings which are transmitted to the housing or frame and to other machine members. Such dynamically unbalanced conditions can occur even though static balance (balance at zero speed) exists. Dynamic balance can be achieved by the addition of one or two masses rotating about the same axis and at the same speed as the unbalanced masses. A single unbalanced mass can be balanced by one counterbalancing mass located 180 degrees opposite and in the same plane of rotation as the unbalanced mass, if the product of their respective radii and masses are equal; i.e., $M_1r_1 = M_2r_2$. Two or more unbalanced masses rotating in the same plane can be balanced by a single mass rotating in the same plane, or by two masses rotating about the same axis in two separate planes. Likewise, two or more unbalanced masses rotating in different planes about a common axis can be balanced by two masses rotating about the same axis in separate planes. When the unbalanced masses are in separate planes they may be in static balance but not in dynamic balance; i.e., they may be balanced when not rotating but unbalanced when rotating. If a system is in dynamic balance, it will remain in balance at all speeds, although this is not strictly true at the critical speed of the system. (See *Critical Speeds*.)

In all the equations that follow, the symbol M denotes either mass in kilograms or in slugs, or weight in pounds. Either mass or weight units may be used and the equations may be used with metric or with customary English units without change; however, in a given problem the units must be all metric or all customary English.

Counterbalancing Several Masses Located in a Single Plane.—In all balancing problems, it is the product of counterbalancing mass (or weight) and its radius that is calculated; it is thus necessary to select either the mass or the radius and then calculate the other value from the product of the two quantities. Design considerations usually make this decision self-evident. The angular position of the counterbalancing mass must also be calculated. Referring to Fig. 1:

$$M_B r_B = \sqrt{(\Sigma Mr\cos\theta)^2 + (\Sigma Mr\sin\theta)^2} \tag{1}$$

$$\tan\theta_B = \frac{-(\Sigma Mr\sin\theta)}{-(\Sigma Mr\cos\theta)} = \frac{y}{x} \tag{2}$$

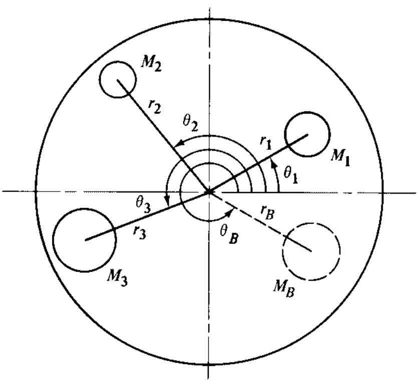
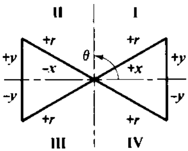


Fig. 1.

Table 1. Relationship of the Signs of the Functions of the Angle with Respect to the Quadrant in Which They Occur



	Angle θ			
	0° to 90°	90° to 180°	180° to 270°	270° to 360°
	Signs of the Functions			
tan	$\frac{+y}{+x}$	$\frac{+y}{-x}$	$\frac{-y}{-x}$	$\frac{-y}{+x}$
sine	$\frac{+y}{+r}$	$\frac{+y}{+r}$	$\frac{-y}{+r}$	$\frac{-y}{+r}$
cosine	$\frac{+x}{+r}$	$\frac{-x}{+r}$	$\frac{-x}{+r}$	$\frac{+x}{+r}$

where:

$M_1, M_2, M_3, \ldots M_n$ = any unbalanced mass or weight, kg or lb

M_B = counterbalancing mass or weight, kg or lb

r = radius to center of gravity of any unbalanced mass or weight, mm or inch

r_B = radius to center of gravity of counterbalancing mass or weight, mm or inch

θ = angular position of r of any unbalanced mass or weight, degrees

θ_B = angular position of r_B of counterbalancing mass or weight, degrees

x and y = see Table 1

Table 1 is helpful in finding the angular position of the counterbalancing mass or weight. It indicates the range of the angles within which this angular position occurs by noting the plus and minus signs of the numerator and the denominator of the terms in Equation (2). In a like manner, Table 1 is helpful in determining the *sign* of the sine or cosine functions for angles ranging from 0 to 360 degrees. Balancing problems are usually solved most conveniently by arranging the arithmetical calculations in a tabular form.

Example: Referring to Fig. 1, the particular values of the unbalanced weights have been entered in the table below. Calculate the magnitude of the counterbalancing weight if its radius is to be 10 inches.

M		*r* in.	θ deg.	cos θ	sin θ	*Mr* cos θ	*Mr* sin θ
No.	lb.						
1	10	10	30	0.8660	0.5000	86.6	50.0
2	5	20	120	−0.5000	0.8660	−50.0	86.6
3	15	15	200	−0.9397	−0.3420	−211.4	−77.0
						−174.8	59.6
						= Σ*Mr* cos θ	= Σ*Mr* sin θ

$$M_B = \frac{\sqrt{(\Sigma Mr\cos\theta)^2 + (\Sigma Mr\sin\theta)^2}}{r_B} = \frac{\sqrt{(-174.8)^2 + (59.6)^2}}{10}$$

$$M_B = 18.5 \text{ lb}$$

$$\tan\theta_B = \frac{-(\Sigma Mr\sin\theta)}{-(\Sigma Mr\cos\theta)} = \frac{-(59.6)}{-(-174.8)} = \frac{-y}{+x}; \quad \theta_B = 341°10'$$

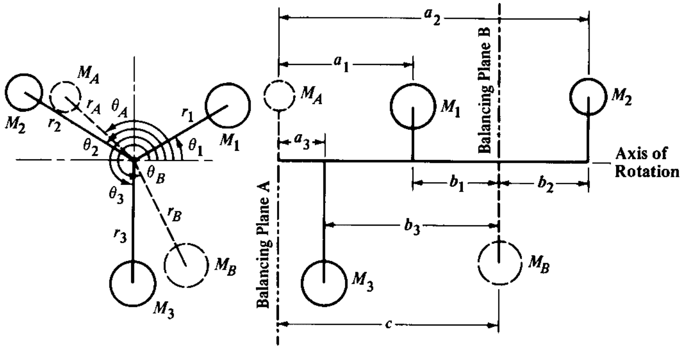


Fig. 2.

Counterbalancing Masses Located in Two or More Planes.—Unbalanced masses or weights rotating about a common axis in two separate planes of rotation form a couple, which must be counterbalanced by masses or weights, also located in two separate planes, call them planes *A* and *B*, and rotating about the same common axis (see *Couples*, page 120). In addition, they must be balanced in the direction perpendicular to the axis, as before. Since two counterbalancing masses are required, two separate equations are required to calculate the product of each mass or weight and its radius, and two additional equations are required to calculate the angular positions. The planes *A* and *B* selected as balancing planes may be any two planes separated by any convenient distance *c*, along the axis of rotation. In Fig. 2:

For balancing plane *A*:

$$M_A r_A = \frac{\sqrt{(\Sigma Mrb\cos\theta)^2 + (\Sigma Mrb\sin\theta)^2}}{c} \qquad (3)$$

$$\tan\theta_A = \frac{-(\Sigma Mrb\sin\theta)}{-(\Sigma Mrb\cos\theta)} = \frac{y}{x} \qquad (4)$$

For balancing plane *B*:

$$M_B r_B = \frac{\sqrt{(\Sigma Mra\cos\theta)^2 + (\Sigma Mra\sin\theta)^2}}{c} \qquad (5)$$

$$\tan\theta_B = \frac{-(\Sigma Mra\sin\theta)}{-(\Sigma Mra\cos\theta)} = \frac{y}{x} \qquad (6)$$

Where: M_A and M_B are the mass or weight of the counterbalancing masses in the balancing planes *A* and *B*, respectively; r_A and r_B are the radii; and θ_A and θ_B are the angular positions of the balancing masses in these planes. *M*, *r*, and θ are the mass or weight, radius, and angular positions of the unbalanced masses, with the subscripts defining the particular mass to which the values are assigned. The length *c*, the distance between the balancing planes, is always a positive value. The axial dimensions, *a* and *b*, may be either positive or negative, depending upon their position relative to the balancing plane; for example, in Fig. 2, the dimension b_2 would be negative.

Example: Referring to Fig. 2, a set of values for the masses and dimensions has been selected and put into convenient table form below. The separation of balancing planes, *c*, is assumed as being 15 inches. If in balancing plane *A*, the radius of the counterbalancing weight is selected to be 10 inches; calculate the magnitude of the counterbalancing mass and its position. If in balancing plane *B*, the counterbalancing mass is selected to be 10 lb; calculate its radius and position.

For balancing plane *A*:

Plane	*M* lb	*r* in.	θ deg.	Balancing Plane *A*			
				b in.	*Mrb*	*Mrb* cos θ	*Mrb* sin θ
1	10	8	30	6	480	415.7	240.0
2	8	10	135	−6	−480	339.4	−339.4
3	12	9	270	12	1296	0.0	−1296.0
A	?	10	?	15[a]	…	755.1	−1395.4
B	10	?	?	0	…	= Σ*Mrb* cos θ	= Σ*Mrb* sin θ

[a] 15 inches = distance *c* between planes *A* and *B*.

$$M_A = \frac{\sqrt{(\Sigma Mrb\cos\theta)^2 + (\Sigma Mrb\sin\theta)^2}}{r_A c} = \frac{\sqrt{(755.1)^2 + (-1395.4)^2}}{10(15)}$$

$$M_A = 10.6 \text{ lb}$$

$$\tan\theta_A = \frac{-(\Sigma Mrb\sin\theta)}{-(\Sigma Mrb\cos\theta)} = \frac{-(-1395.4)}{-(755.1)} = \frac{+y}{-x}$$

$$\theta_A = 118°25'$$

For balancing plane *B*:

Plane	*M* lb	*r* in.	θ deg.	Balancing Plane *B*			
				a in.	*Mra*	*Mra* cos θ	*Mra* sin θ
1	10	8	30	9	720	623.5	360.0
2	8	10	135	21	1680	−1187.9	1187.9
3	12	9	270	3	324	0.0	−324.0
A	?	10	?	0	…	−564.4	1223.9
B	10	?	?	15[a]	…	=Σ*Mra* cos θ	=Σ*Mra* sin θ

[a] 15 inches = distance *c* between planes *A* and *B*.

$$r_B = \frac{\sqrt{(\Sigma Mra\cos\theta)^2 + (\Sigma Mra\sin\theta)^2}}{M_B c} = \frac{\sqrt{(-564.4)^2 + (1223.9)^2}}{10(15)}$$

$$= 8.985 \text{ in.}$$

$$\tan\theta_B = \frac{-(\Sigma Mra\sin\theta)}{-(\Sigma Mra\cos\theta)} = \frac{-(1223.9)}{-(-564.4)} = \frac{-y}{+x}$$

$$\theta_B = 294°45'$$

Balancing Lathe Fixtures.—Lathe fixtures rotating at a high speed require balancing. Often it is assumed that the center of gravity of the workpiece and fixture, and of the counterbalancing masses are in the same plane; however, this is not usually the case. Counterbalancing masses are required in two separate planes to prevent excessive vibration or bearing loads at high speeds.

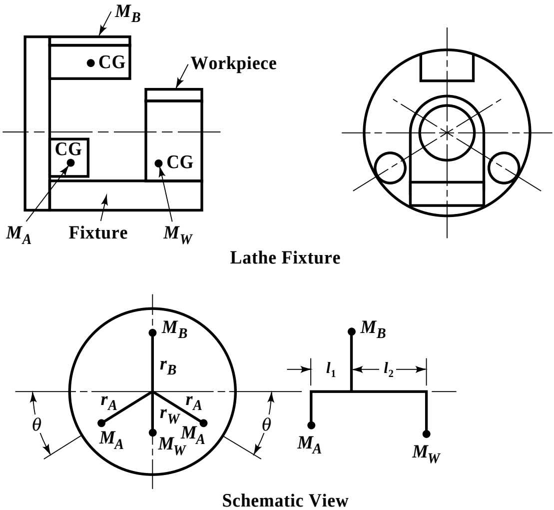


Fig. 3.

Usually a single counterbalancing mass is placed in one plane selected to be 180 degrees directly opposite the combined center of gravity of the workpiece and the fixture. Two equal counterbalancing masses are then placed in the second counterbalancing plane, equally spaced on each side of the fixture. Referring to Fig. 3, the two counterbalancing masses M_A and the two angles θ are equal. For the design in this illustration, the following formulas can be used to calculate the magnitude of the counterbalancing masses. Since their angular positions are fixed by the design, they are not calculated.

$$M_B = \frac{M_w r_w (l_1 + l_2)}{r_B l_1} \tag{7}$$

$$M_A = \frac{M_B r_B - M_w r_w}{2 r_A \sin\theta} \tag{8}$$

In these formulas M_w and r_w denote the mass or weight and the radius of the combined center of gravity of the workpiece and the fixture.

In Fig. 3 the combined weight of the workpiece and the fixture is 18.5 lb. The following dimensions were determined from the layout of the fixture and by calculating the centers of gravity: $r_w = 2$ in.; $r_A = 6.25$ in.; $r_B = 6$ in.; $l_1 = 3$ in.; $l_2 = 5$ in.; and $\theta = 30°$. Calculate the weights of the counterbalancing masses.

$$M_B = \frac{M_w r_w (l_1 + l_2)}{r_B l_1} = \frac{18.5 \times 2 \times 8}{6 \times 3} = 16.44 \text{ lb}$$

$$M_A = \frac{M_B r_B - M_w r_w}{2 r_A \sin\theta} = \frac{(16.44 \times 6) - (18.5 \times 2)}{(2 \times 6.25)\sin 30^\circ} = 9.86 \text{ lb (each weight)}$$

FLYWHEELS

Classification of Flywheels.—Flywheels may be classified as *balance wheels* or as *flywheel pulleys*. The object of all flywheels is to equalize the energy exerted and the work done and thereby prevent excessive or sudden changes of speed. The permissible speed variation is an important factor in all flywheel designs. The allowable speed change varies considerably for different classes of machinery; for instance, it is about 1 or 2 per cent in steam engines, while in punching and shearing machinery a speed variation of 20 per cent may be allowed.

The function of a balance wheel is to absorb and equalize energy in case the resistance to motion, or driving power, varies throughout the cycle. Therefore, the rim section is generally quite heavy and is designed with reference to the energy that must be stored in it to prevent excessive speed variations and, with reference to the strength necessary to withstand safely the stresses resulting from the required speed. The rims of most balance wheels are either square or nearly square in section, but flywheel pulleys are commonly made wide to accommodate a belt and relatively thin in a radial direction, although this is not an invariable rule.

Flywheels, in general, may either be formed of a solid or one-piece section, or they may be of sectional construction. Flywheels in diameters up to about eight feet are usually cast solid, the hubs sometimes being divided to relieve cooling stresses. Flywheels ranging from, say, eight feet to fifteen feet in diameter, are commonly cast in half sections, and the larger sizes in several sections, the number of which may equal the number of arms in the wheel. Sectional flywheels may be divided into two general classes. One class includes cast wheels which are formed of sections principally because a solid casting would be too large to transport readily. The second class includes wheels of sectional construction which, by reason of the materials used and the special arrangement of the sections, enables much higher peripheral speeds to be obtained safely than would be possible with ordinary sectional wheels of the type not designed especially for high speeds. Various designs have been built to withstand the extreme stresses encountered in some classes of service. The rims in some designs are laminated, being partly or entirely formed of numerous segment-shaped steel plates. Another type of flywheel, which is superior to an ordinary sectional wheel, has a solid cast-iron rim connected to the hub by disk-shaped steel plates instead of cast spokes.

Steel wheels may be divided into three distinct types, including 1) those having the center and rim built up entirely of steel plates; 2) those having a cast-iron center and steel rim; and 3) those having a cast-steel center and rim formed of steel plates.

Wheels having wire-wound rims have been used to a limited extent when extremely high speeds have been necessary.

When the rim is formed of sections held together by joints it is very important to design these joints properly. The ordinary bolted and flanged rim joints located between the arms average about 20 per cent of the strength of a solid rim and about 25 per cent is the maximum strength obtainable for a joint of this kind. However, by placing the joints at the ends of the arms instead of between them, an efficiency of 50 per cent of the strength of the rim may be obtained, because the joint is not subjected to the outward bending stresses between the arms but is directly supported by the arm, the end of which is secured to the rim just beneath the joint. When the rim sections of heavy balance wheels are held together by steel links shrunk into place, an efficiency of 60 per cent may be obtained; and by using a rim of box or I-section, a link type of joint connection may have an efficiency of 100 percent.

Flywheel Calculations

Energy Due to Changes of Velocity.—When a flywheel absorbs energy from a variable driving force, as in a steam engine, the velocity increases; and when this stored energy is given out, the velocity diminishes. When the driven member of a machine encounters a variable resistance in performing its work, as when the punch of a punching machine is passing through a steel plate, the flywheel gives up energy while the punch is at work, and, consequently, the speed of the flywheel is reduced. The total energy that a flywheel would give out if brought to a standstill is given by the formula:

$$E = \frac{Wv^2}{2g} = \frac{Wv^2}{64.32}$$

in which E = total energy of flywheel, in foot-pounds
W = weight of flywheel rim, in pounds
v = velocity at mean radius of flywheel rim, in feet per second
g = acceleration due to gravity = 32.16 ft/s^2

If the velocity of a flywheel changes, the energy it will absorb or give up is proportional to the difference between the squares of its initial and final speeds, and is equal to the difference between the energy that it would give out if brought to a full stop and the energy that is still stored in it at the reduced velocity. Hence:

$$E_1 = \frac{Wv_1^2}{2g} - \frac{Wv_2^2}{2g} = \frac{W(v_1^2 - v_2^2)}{64.32}$$

in which E_1 = energy in foot-pounds that a flywheel will give out while the speed is reduced from v_1 to v_2
W = weight of flywheel rim, in pounds
v_1 = velocity at mean radius of flywheel rim before any energy has been given out, in feet per second
v_2 = velocity of flywheel rim at end of period during which the energy has been given out, in feet per second

Ordinarily, the effects of the arms and hub do not enter into flywheel calculations, and only the weight of the rim is considered. In computing the velocity, the mean radius of the rim is commonly used.

Using metric SI units, the formulas are $E = ½Mv^2$, and $E_1 = ½M(v_1^2 - v_2^2)$, where E and E_1 are in joules; M = the mass of the rim in kilograms; and v, v_1, and v_2 = velocities in meters per second. Note: In the SI, the unit of mass is the kilogram. If the weight of the flywheel rim is given in kilograms, the value referred to is the mass, *M*. Should the weight be given in newtons, *N*, then

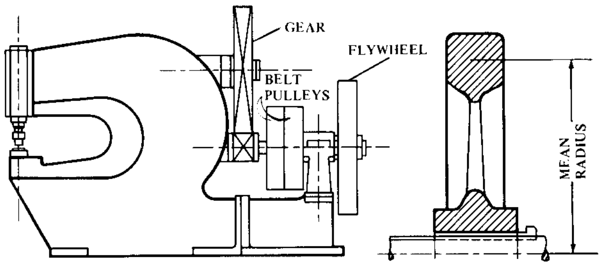
$$M = \frac{W(\text{newtons})}{g}$$

where *g* is approximately 9.81 meters per second squared.

General Procedure in Flywheel Design.—The general method of designing a flywheel is to determine first the value of E_1 or the energy the flywheel must either supply or absorb for a given change in velocity, which, in turn, varies for different classes of service. The mean diameter of the flywheel may be assumed, or it may be fixed within certain limits by the general design of the machine. Ordinarily the speed of the flywheel shaft is known, at least approximately; the values of v_1 and v_2 can then be determined, the latter depending upon the allowable percentage of speed variation. When these values are known, the weight of the rim and the cross-sectional area required to obtain this weight may be computed. The general procedure will be illustrated more in detail by considering the design of flywheels for punching and shearing machinery.

Flywheels for Presses, Punches, Shears, Etc.—In these classes of machinery, the work that the machine performs is of an intermittent nature and is done during a small part of the time required for the driving shaft of the machine to make a complete revolution. To distribute the work of the machine over the entire period of revolution of the driving shaft, a heavy-rimmed flywheel is placed on the shaft, giving the belt an opportunity to perform an almost uniform amount of work during the whole revolution. During the greater part of the revolution of the driving shaft, the belt power is used to accelerate the speed of the flywheel. During the part of the revolution when the work is done, the energy thus stored up in the flywheel is given out at the expense of its velocity. The problem is to determine the weight and cross-sectional area of the rim when the conditions affecting the design of the flywheel are known.

Example: A flywheel is required for a punching machine capable of punching ¾-inch holes through structural steel plates ¾ inch thick. This machine (see accompanying diagram) is of the general type having a belt-driven shaft at the rear which carries a flywheel and a pinion that meshes with a large gear on the main shaft at the top of the machine. It is assumed that the relative speeds of the pinion and large gear are 7 to 1, respectively, and that the slide is to make 30 working strokes per minute. The preliminary layout shows that the flywheel should have a mean diameter (see enlarged detail) of about 30 inches. Find the weight of the flywheel and the remaining rim dimensions.



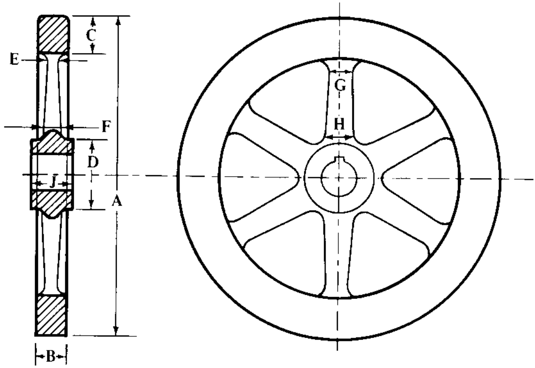
Punch Presss and Flywheel Detail

Energy Supplied by Flywheel: The energy that the flywheel must give up for a given change in velocity, and the weight of rim necessary to supply that energy, must be determined. The maximum force for shearing a ¾-inch hole through ¾-inch structural steel equals approximately the circumference of the hole multiplied by the thickness of the stock multiplied by the tensile strength, which is nearly the same as the shearing resistance of the steel. Thus, in this case, 3.1416 × ¾ × ¾ × 60,000 = 106,000 pounds. The average force will be much less than the maximum. Some designers assume that the average force is about one-half the maximum, although experiments show that the material is practically sheared off when the punch has entered the sheet a distance equal to about one-third the sheet thickness. On this latter basis, the average energy E_a is 2200 foot-pounds for the example given. Thus:

$$E_a = \frac{106,000 \times \frac{1}{3} \times \frac{3}{4}}{12} = \frac{106,000}{4 \times 12} = 2200 \text{ foot-pounds.}$$

If the efficiency of the machine is taken as 85 per cent, the energy required will equal 2200 0.85 = 2600 foot-pounds nearly. Assume that the energy supplied by the belt while the punch is at work is determined by calculation to equal 175 foot-pounds. Then the flywheel must supply 2600 - 175 = 2425 foot-pounds = E_1.

Dimensions of Flywheels for Punches and Shears



A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*J*	Max. R.P.M.
24	3	3½	6	1¼	1⅜	2¾	3¼	3½	955
30	3½	4	7	1⅜	1½	3	3¾	4	796
36	4	4½	8	1½	1¾	3¼	4¼	4½	637
42	4¼	4¾	9	1¾	2	3½	4½	5	557
48	4½	5	10	1¾	2	3¾	4¾	5½	478
54	4¾	5½	11	2	2¼	4	5	6	430
60	5	6	12	2¼	2½	4½	5½	6½	382
72	5½	7	13	2½	2¾	5	6½	7	318
84	6	8	14	3	3½	5½	7½	8	273
96	7	9	15	3½	4	6	9	9	239
108	8	10	16½	3¾	4½	6½	10½	10	212
120	9	11	18	4	5	7½	12	12	191

The maximum number of revolutions per minute given in this table should never be exceeded for cast-iron flywheels.

Rim Velocity at Mean Radius: When the mean radius of the flywheel is known, the velocity of the rim at the mean radius, in feet per second, is:

$$v = \frac{2 \times 3.1416 \times R \times n}{60}$$

in which v = velocity at mean radius of flywheel, in feet per second

R = mean radius of flywheel rim, in feet

n = number of revolutions per minute

According to the preliminary layout the mean diameter in this example should be about 30 inches and the driving shaft is to make 210 rpm, hence,

$$v = \frac{2 \times 3.1416 \times 1.25 \times 210}{60} = 27.5 \text{ feet per second}$$

Weight of Flywheel Rim: Assuming that the allowable variation in velocity when punching is about 15 per cent, and values of v_1 and v_2 are respectively 27.5 and 23.4 feet per second ($27.5 \times 0.85 = 23.4$), the weight of a flywheel rim necessary to supply a given amount of energy in foot-pounds while the speed is reduced from v_1 to v_2 would be:

$$W = \frac{E_1 \times 64.32}{v_1^2 - v_2^2} = \frac{2425 \times 64.32}{27.5^2 - 23.4^2} = 750 \text{ pounds}$$

Size of Rim for Given Weight: Since 1 cubic inch of cast iron weighs 0.26 pound, a flywheel rim weighing 750 pounds contains 750 0.26 = 2884 cubic inches. The cross-sectional area of the rim in square inches equals the total number of cubic inches divided by the mean circumference, or 2884 94.25 = 31 square inches nearly, which is approximately the area of a rim 5⅛ inches wide and 6 inches deep.

Simplified Flywheel Calculations.—Calculations for designing the flywheels of punches and shears are simplified by the following formulas and the accompanying table of constants applying to different percentages of speed reduction. In these formulas let:

HP = horsepower required
N = number of strokes per minute
E = total energy required per stroke, in foot-pounds
E_1 = energy given up by flywheel, in foot-pounds
T = time in seconds per stroke
T_1 = time in seconds of actual cut
W = weight of flywheel rim, in pounds
D = mean diameter of flywheel rim, in feet
R = maximum allowable speed of flywheel in revolutions per minute
C and C_1 = speed reduction values as given in table
a = width of flywheel rim
b = depth of flywheel rim
y = ratio of depth to width of rim

$$HP = \frac{EN}{33,000} = \frac{E}{T \times 550} \qquad E_1 = E\left(1 - \frac{T_1}{T}\right)$$

$$W = \frac{E_1}{CD^2R^2} \qquad a = \sqrt{\frac{1.22W}{12Dy}} \qquad b = ay$$

For cast-iron flywheels, with a maximum stress of 1000 pounds per square inch:

$$W = C_1E_1 \qquad R = 1940 \div D$$

Values of C and C_1 in the Previous Formulas

Per Cent Reduction	C	C_1	Per Cent Reduction	C	C_1
2½	0.00000213	0.1250	10	0.00000810	0.0328
5	0.00000426	0.0625	15	0.00001180	0.0225
7½	0.00000617	0.0432	20	0.00001535	0.0173

Example 1: A hot slab shear is required to cut a slab 4×15 inches which, at a shearing stress of 6000 pounds per square inch, gives a force between the knives of 360,000 pounds. The total energy required for the cut will then be $360{,}000 \times 4/12 = 120{,}000$ foot-pounds. The shear is to make 20 strokes per minute; the actual cutting time is 0.75 second, and the balance of the stroke is 2.25 seconds.

The flywheel is to have a mean diameter of 6 feet 6 inches and is to run at a speed of 200 rpm; the reduction in speed to be 10 per cent per stroke when cutting.

$$HP = \frac{120,000 \times 20}{33,000} = 72.7 \text{ horsepower}$$

$$E_1 = 120,000 \times \left(1 - \frac{0.75}{3}\right) = 90,000 \text{ foot-pounds}$$

$$W = \frac{90,000}{0.0000081 \times 6.5^2 \times 200^2} = 6570 \text{ pounds}$$

Assuming a ratio of 1.22 between depth and width of rim,

$$a = \sqrt{\frac{6570}{12 \times 6.5}} = 9.18 \text{ inches}$$

$$b = 1.22 \times 9.18 = 11.2 \text{ inches}$$

or size of rim, say, $9 \times 11\frac{1}{2}$ inches.

Example 2: Suppose that the flywheel in Example 1 is to be made with a stress due to centrifugal force of 1000 pounds per square inch of rim section.

$$C_1 \text{ for 10 per cent} = 0.0328$$

$$W = 0.0328 \times 90,000 = 2950 \text{ pounds}$$

$$R = \frac{1940}{D} \qquad \text{If } D = 6 \text{ feet,} \qquad R = \frac{1940}{6} = 323 \text{ rpm}$$

Assuming a ratio of 1.22 between depth and width of rim, as before:

$$a = \sqrt{\frac{2950}{12 \times 6}} = 6.4 \text{ inches}$$

$$b = 1.22 \times 6.4 = 7.8 \text{ inches}$$

or size of rim, say, $6\frac{1}{4} \times 8$ inches.

Centrifugal Stresses in Flywheel Rims.—In general, high speed is desirable for flywheels in order to avoid using wheels that are unnecessarily large and heavy. The centrifugal tension or hoop tension stress, that tends to rupture a flywheel rim of given area, depends solely upon the rim velocity and is independent of the rim radius. The bursting velocity of a flywheel, based on hoop stress alone (not considering bending stresses), is related to the tensile stress in the flywheel rim by the following formula which is based on the centrifugal force formula from mechanics.

$$V = \sqrt{10 \times s} \qquad \text{or,} \qquad s = V^2 \div 10$$

where V = velocity of outside circumference of rim in feet per second, and s is the tensile strength of the rim material in pounds per square inch.

For cast iron having a tensile strength of 19,000 pounds per square inch the bursting speed would be:

$$V = \sqrt{10 \times 19,000} = 436 \text{ feet per second}$$

Built-up Flywheels: Flywheels built up of solid disks of rolled steel plate stacked and bolted together on a through shaft have greater speed capacity than other types. The maximum hoop stress is at the bore and is given by the formula,

$$s = 0.0194V^2[4.333 + (d/D)^2]$$

In this formula, s and V are the stress and velocity as previously defined and d and D are the bore and outside diameters, respectively.

Assuming the plates to be of steel having a tensile strength of 60,000 pounds per square inch and a safe working stress of 24,000 pounds per square inch (using a factor of safety of 2.5 on stress or $\sqrt{2.5}$ on speed) and taking the worst condition (when d approaches D), the safe rim speed for this type of flywheel is 500 feet per second or 30,000 feet per minute.

Combined Stresses in Flywheels.—The bending stresses in the rim of a flywheel may exceed the centrifugal (hoop tension) stress predicted by the simple formula $s = V^2$ 10 by a considerable amount. By taking into account certain characteristics of flywheels, relatively simple formulas have been developed to determine the stress due to the combined effect of hoop tension and bending stress. Some of the factors that influence the magnitude of the maximum combined stress acting at the rim of a flywheel are:

1) *The number of spokes*. Increasing the number of spokes decreases the rim span between spokes and hence decreases the bending moment. Thus an eight-spoke wheel can be driven to a considerably higher speed before bursting than a six-spoke wheel having the same rim.

2) *The relative thickness of the spokes.* If the spokes were extremely thin, like wires, they could offer little constraint to the rim in expanding to its natural diameter under centrifugal force, and hence would cause little bending stress. Conversely, if the spokes were extremely heavy in proportion to the rim, they would restrain the rim thereby setting up heavy bending stresses at the junctions of the rim and spokes.

3) *The relative thickness of the rim to the diameter.* If the rim is quite thick (i.e., has a large section modulus in proportion to span), its resistance to bending will be great and bending stress small. Conversely, thin rims with a section modulus small in comparison with diameter or span have little resistance to bending, thus are subject to high bending stresses.

4) *Residual stresses*. These include shrinkage stresses, impact stresses, and stresses caused by operating torques and imperfections in the material. Residual stresses are taken into account by the use of a suitable factor of safety. (See *Factors of Safety for Flywheels.*)

The formulas that follow give the maximum combined stress at the rim of fly-wheels having 6, 8, and 10 spokes. These formulas are for flywheels with *rectangular rim sections* and take into account the first three of the four factors listed as influencing the magnitude of the combined stress in flywheels.

For 6 spokes:
$$s = \frac{V^2}{10}\left[1 + \left(\frac{0.56B - 1.81}{3Q + 3.14}\right)Q\right]$$

For 8 spokes:
$$s = \frac{V^2}{10}\left[1 + \left(\frac{0.42B - 2.53}{4Q + 3.14}\right)Q\right]$$

For 10 spokes:
$$s = \frac{V^2}{10}\left[1 + \left(\frac{0.33B - 3.22}{5Q + 3.14}\right)Q\right]$$

In these formulas, s = maximum combined stress in pounds per square inch; Q = ratio of mean spoke cross-section area to rim cross-section area; B = ratio of outside diameter of rim to rim thickness; and V = velocity of flywheel rim in feet per second.

Thickness of Cast Iron Flywheel Rims.—The mathematical analysis of the stresses in flywheel rims is not conclusive owing to the uncertainty of shrinkage stresses in castings or the strength of the joint in sectional wheels. When a flywheel of ordinary design is revolving at high speed, the tendency of the rim is to bend or bow outward between the arms, and the bending stresses may be serious, especially if the rim is wide and thin and the spokes are rather widely spaced. When the rims are thick, this tendency does not need to be considered, but in a thin rim running at high speed, the stress in the middle might become sufficiently great to cause the wheel to fail. The proper thickness of a cast-iron rim to resist this tendency is given for solid rims by Formula (1) and for a jointed rim by Formula (2).

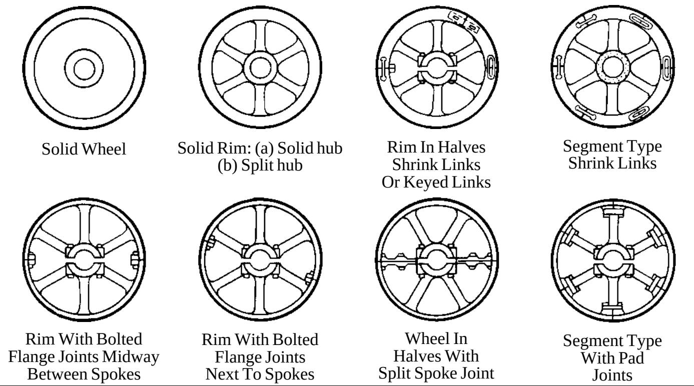
$$t = \frac{0.475d}{n^2\left(\frac{6000}{v^2} - \frac{1}{10}\right)} \qquad (1) \qquad\qquad t = \frac{0.95d}{n^2\left(\frac{6000}{v^2} - \frac{1}{10}\right)} \qquad (2)$$

In these formulas, t = thickness of rim, in inches; d = diameter of flywheel, in inches; n = number of arms; v = peripheral speed, in feet per second.

Factors of Safety for Flywheels.—Cast-iron flywheels are commonly designed with a factor of safety of 10 to 13. A factor of safety of 10 applied to the tensile strength of a flywheel material is equivalent to a factor of safety of $\sqrt{10}$ or 3.16 on the speed of the flywheel because the stress on the rim of a flywheel increases as thdesigned would undergo rim stresses four times as great as at the design speed.

Tables of Safe Speeds for Flywheels.—The accompanying Table 1, prepared by T. C. Rathbone of The Fidelity and Casualty Company of New York, gives general recommendations for safe rim speeds for flywheels of various constructions. Table 2 shows the number of revolutions per minute corresponding to the rim speeds in Table 1.

Table 1. Safe Rim Speeds for Flywheels



Type of Wheel	Safe Rim Speed	
	Feet per Sec.	Feet per Min.
Solid cast iron (balance wheels—heavy rims)	110	6,600
Solid cast iron (pulley wheels—thin rims)	85	5,100
Wheels with shrink link joints	77.5	4,650
Wheels with pad type joints	70.7	4,240
Wheels with bolted flange joints	50	3,000
Solid cast steel wheels	200	12,000
Wheels built up of stacked steel plates	500	30,000

To find the safe speed in revolutions per minute, divide the safe rim speed in feet per minute by 3.14 times the outside diameter of the flywheel rim in feet. For flywheels up to 15 feet in diameter, see Table 2.

Table 2. Safe Speeds of Rotation for Flywheels

Outside Diameter of Rim (feet)	Safe Rim Speed in Feet per Minute (from Table 1)						
	6,600	5,100	4,650	4,240	3,000	12,000	30,000
	Safe Speed of Rotation in Revolutions per Minute						
1	2100	1623	1480	1350	955	3820	9549
2	1050	812	740	676	478	1910	4775
3	700	541	493	450	318	1273	3183
4	525	406	370	338	239	955	2387
5	420	325	296	270	191	764	1910
6	350	271	247	225	159	637	1592
7	300	232	211	193	136	546	1364
8	263	203	185	169	119	478	1194
9	233	180	164	150	106	424	1061
10	210	162	148	135	96	382	955
11	191	148	135	123	87	347	868
12	175	135	123	113	80	318	796
13	162	125	114	104	73	294	735
14	150	116	106	97	68	273	682
15	140	108	99	90	64	255	637

Safe speeds of rotation are based on safe rim speeds shown in Table 1.

Safe Speed Formulas for Flywheels and Pulleys.—No simple formula can accommodate all the various types and proportions of flywheels and pulleys and at the same time provide a uniform factor of safety for each. Because of considerations of safety, such a formula would penalize the better constructions to accommodate the weaker designs.

One formula that has been used to check the maximum rated operating speed of flywheels and pulleys and which takes into account material properties, construction, rim thickness, and joint efficiencies is the following:

$$N = \frac{CAMEK}{D}$$

In this formula,

N = maximum rated operating speed in revolutions per minute

C = 1.0 for wheels driven by a constant speed electric motor (i.e., a-c squirrel-cage induction motor or a-c synchronous motor, etc.)
0.90 for wheels driven by variable speed motors, engines or turbines where overspeed is not over 110 per cent of rated operating speed

A = 0.90 for 4 arms or spokes
1.00 for 6 arms or spokes
1.08 for 8 arms or spokes
1.50 for disc type

M = 1.00 for cast iron of 20,000 psi tensile strength, or unknown
1.12 for cast iron of 25,000 psi tensile strength
1.22 for cast iron of 30,000 psi tensile strength
1.32 for cast iron of 35,000 psi tensile strength
2.20 for nodular iron of 60,000 psi tensile strength
2.45 for cast steel of 60,000 psi tensile strength
2.75 for plate or forged steel of 60,000 psi tensile strength

E = joint efficiency
1.0 for solid rim
0.85 for link or prison joints
0.75 for split rim — bolted joint at arms

0.70 for split rim — bolted joint between arms

K = 1355 for rim thickness equal to 1 per cent of outside diameter
1650 for rim thickness equal to 2 per cent of outside diameter
1840 for rim thickness equal to 3 per cent of outside diameter
1960 for rim thickness equal to 4 per cent of outside diameter
2040 for rim thickness equal to 5 per cent of outside diameter
2140 for rim thickness equal to 7 per cent of outside diameter
2225 for rim thickness equal to 10 per cent of outside diameter
2310 for rim thickness equal to 15 per cent of outside diameter
2340 for rim thickness equal to 20 per cent of outside diameter

D = outside diameter of rim in feet

A six-spoke solid cast iron balance wheel 8 feet in diameter has a rectangular rim 10 inches thick. What is the safe speed, in revolutions per minute, if driven by a constant speed motor?

In this instance, $C = 1$; $A = 1$; $M = 1$, since tensile strength is unknown; $E = 1$; $K = 2225$ since the rim thickness is approximately 10 per cent of the wheel diameter; and $D = 8$ feet. Thus,

$$N = \frac{1 \times 1 \times 1 \times 2225}{8} = 278 \text{ rpm}$$

(*Note:* This safe speed is slightly greater than the value of 263 rpm obtainable directly from Tables 1 and 2.)

Tests to Determine Flywheel Bursting Speeds.—Tests made by Prof. C. H. Benjamin, to determine the bursting speeds of flywheels, showed the following results:

Cast-iron Wheels with Solid Rims: Cast-iron wheels having solid rims burst at a rim speed of 395 feet per second, corresponding to a centrifugal tension of about 15,600 pounds per square inch.

Wheels with Jointed Rims: Four wheels were tested with joints and bolts inside the rim, using the familiar design ordinarily employed for band wheels, but with the joints located at points one-fourth of the distance from one arm to the next. These locations represent the points of least bending moment, and, consequently, the points at which the deflection due to centrifugal force would be expected to have the least effect. The tests, however, did not bear out this conclusion. The wheels burst at a rim speed of 194 feet per second, corresponding to a centrifugal tension of about 3750 pounds per square inch. These wheels, therefore, were only about one-quarter as strong as the wheels with solid rims, and burst at practically the same speed as wheels in a previous series of tests in which the rim joints were midway between the arms.

Bursting Speed for Link Joints: Another type of wheel with deep rim, fastened together at the joints midway between the arms by links shrunk into recesses, after the manner of flywheels for massive engines, gave much superior results. This wheel burst at a speed of 256 feet per second, indicating a centrifugal tension of about 6600 pounds per square inch.

Wheel having Tie-rods: Tests were made on a band wheel having joints inside the rim, midway between the arms, and in all respects like others of this design previously tested, except that tie-rods were used to connect the joints with the hub. This wheel burst at a speed of 225 feet per second, showing an increase of strength of from 30 to 40 per cent over similar wheels without the tie-rods.

Wheel Rim of I-section: Several wheels of special design, not in common use, were also tested, the one giving the greatest strength being an English wheel, with solid rim of I-section, made of high-grade cast iron and with the rim tied to the hub by steel wire spokes. These spokes were adjusted to have a uniform tension. The wheel gave way at a rim speed of 424 feet per second, which is slightly higher than the speed of rupture of the solid rim wheels with ordinary style of spokes.

Tests on Flywheel of Special Construction: A test was made on a flywheel 49 inches in diameter and weighing about 900 pounds. The rim was 6¾ inches wide and 1⅛ inches thick, and was built of ten segments, the material being cast steel. Each joint was secured by three "prisoners" of an I-section on the outside face, by link prisoners on each edge, and by a dovetailed bronze clamp on the inside, fitting over lugs on the rim. The arms were of phosphor-bronze, twenty in number, ten on each side, and were cros-shaped in section. These arms came midway between the rim joints and were bolted to plane faces on the polygonal hub. The rim was further reinforced by a system of diagonal bracing, each section of the rim being supported at five points on each side, in such a way as to relieve it almost entirely from bending. The braces, like the arms, were of phosphor-bronze, and all bolts and connecting links were of steel. This wheel was designed as a model of a proposed 30-foot flywheel. On account of the excessive air resistance the wheel was enclosed at the sides between sheet-metal disks. This wheel burst at 1775 revolutions per minute or at a linear speed of 372 feet per second. The hub and main spokes of the wheel remained nearly in place, but parts of the rim were found 200 feet away. This sudden failure of the rim casting was unexpected, as it was thought the flange bolts would be the parts to give way first. The tensile strength of the casting at the point of fracture was about four times the strength of the wheel rim at a solid section.

Stresses in Rotating Disks.—When a disk of uniform width is rotated, the maximum stress S_t is tangential and at the bore of the hub, and the tangential stress is always greater than the radial stress at the same point on the disk. If S_t = maximum tangential stress in pounds per sq. in.; w = weight of material, lb. per cu. in.; N = rev. per min.; m = Poisson's ratio = 0.3 for steel; R = outer radius of disk, inches; r = inner radius of disk or radius of bore, inches.

$$S_t = 0.000071wN^2[(3+m)R^2 + (1-m)r^2]$$

Steam Engine Flywheels.—The variable amount of energy during each stroke and the allowable percentage of speed variation are of special importance in designing steam engine flywheels. The earlier the point of cut-off, the greater the variation in energy and the larger the flywheel that will be required. The weight of the reciprocating parts and the length of the connecting-rod also affect the variation. The following formula is used for computing the weight of the flywheel rim:

Let W = weight of rim in pounds

D = mean diameter of rim in feet

N = number of revolutions per minute

$1/n$ = allowable variation in speed (from 1/50 to 1/100)

E = excess and deficiency of energy in foot-pounds

c = factor of energy excess, from the accompanying table

HP = indicated horsepower

Then, if the indicated horsepower is given:

$$W = \frac{387{,}587{,}500 \times cn \times \mathrm{HP}}{D^2N^3} \qquad (1)$$

If the work in foot-pounds is given, then:

$$W = \frac{11,745nE}{D^2N^2} \qquad (2)$$

In the second formula, E equals the average work in foot-pounds done by the engine in one revolution, multiplied by the decimal given in the accompanying table, "*Factors for Engine Flywheel Calculations*," which covers both condensing and non-condensing engines:

Factors for Engine Flywheel Calculations

Condensing Engines						
Fraction of stroke at which steam is cut off	1§3	1§4	1§5	1§6	1§7	1§8
Factor of energy excess	0.163	0.173	0.178	0.184	0.189	0.191
Non-condensing Engines						
Steam cut off at			1§2	1§3	1§4	1§5
Factor of energy excess			0.160	0.186	0.209	0.232

Example 1: A non-condensing engine of 150 indicated horsepower is to make 200 revolutions per minute, with a speed variation of 2 per cent. The average cut-off is to be at one-quarter stroke, and the flywheel is to have a mean diameter of 6 feet. Find the necessary weight of the rim in pounds.

From the table $c = 0.209$, and from the data given HP = 150; $N = 200$; $1/n = 1/50$ or $n = 50$; and, $D = 6$.

Substituting these values in Equation (1):

$$W = \frac{387,587,500 \times 0.209 \times 50 \times 150}{6^2 \times 200^3} = 2110 \text{ pounds, nearly}$$

Example 2: A condensing engine, 24 × 42 inches, cuts off at one-third stroke and has a mean effective pressure of 50 pounds per square inch. The flywheel is to be 18 feet in mean diameter and make 75 revolutions per minute with a variation of 1 per cent. Find the required weight of the rim.

The work done on the piston in one revolution is equal to the pressure on the piston multiplied by the distance traveled or twice the stroke in feet. The area of the piston is 452.4 square inches, and twice the stroke is 7 feet. The work done on the piston in one revolution is, therefore, 452.4 × 50 × 7 = 158,340 foot-pounds. From the table $c = 0.163$, and therefore:

$$E = 158,340 \times 0.163 = 25,810 \text{ foot-pounds}$$

From the data given: $n = 100$; $D = 18$; $N = 75$. Substituting these values in Equation (2):

$$W = \frac{11,745 \times 100 \times 25,810}{18^2 \times 75^2} = 16,650 \text{ pounds, nearly}$$

Spokes or Arms of Flywheels.—Flywheel arms are usually of elliptical cross-section. The major axis of the ellipse is in the plane of rotation to give the arms greater resistance to bending stresses and reduce the air resistance which may be considerable at high velocity. The stresses in the arms may be severe, due to the inertia of a heavy rim when sudden load changes occur. The strength of the arms should equal three-fourths the strength of the shaft in torsion.

If W equals the width of the arm at the hub (length of major axis) and D equals the shaft diameter, then W equals 1.3 D for a wheel having 6 arms; and for an 8-arm wheel W equals 1.2 D. The thickness of the arm at the hub (length of minor axis) equals one-half the width. The arms usually taper toward the rim. The cross-sectional area at the rim should not be less than two-thirds the area at the hub.

Critical Speeds

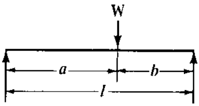
Critical Speeds of Rotating Bodies and Shafts.—If a body or disk mounted upon a shaft rotates about it, the center of gravity of the body or disk must be at the center of the shaft, if a perfect running balance is to be obtained. In most cases, however, the center of gravity of the disk will be slightly removed from the center of the shaft, owing to the difficulty of perfect balancing. Now, if the shaft and disk be rotated, the centrifugal force generated by the heavier side will be greater than that generated by the lighter side geometrically opposite to it, and the shaft will deflect toward the heavier side, causing the center of the disk to rotate in a small circle. A rotating shaft without a body or disk mounted on it can also become dynamically unstable, and the resulting vibrations and deflections can result in damage not only to the shaft but to the machine of which it is a part. These conditions hold true up to a comparatively high speed; but a point is eventually reached (at several thousand revolutions per minute) when momentarily there will be excessive vibration, and then the parts will run quietly again. The speed at which this occurs is called the *critical speed* of the wheel or shaft, and the phenomenon itself for the shaft-mounted disk or body is called the *settling* of the wheel. The explanation of the settling is that at this speed the axis of rotation changes, and the wheel and shaft, instead of rotating about their geometrical center, begin to rotate about an axis through their center of gravity. The shaft itself is then deflected so that for every revolution its geometrical center traces a circle around the center of gravity of the rotating mass.

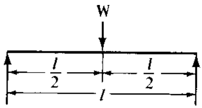
Critical speeds depend upon the magnitude or location of the load or loads carried by the shaft, the length of the shaft, its diameter and the kind of supporting bearings. The normal operating speed of a machine may or may not be higher than the critical speed. For instance, some steam turbines exceed the critical speed, although they do not run long enough at the critical speed for the vibrations to build up to an excessive amplitude. The practice of the General Electric Co. at Schenectady is to keep below the critical speeds. It is assumed that the maximum speed of a machine may be within 20 per cent high or low of the critical speed without vibration troubles. Thus, in a design of steam turbine sets, critical speed is a factor that determines the size of the shafts for both the generators and turbines. Although a machine may run very close to the critical speed, the alignment and play of the bearings, the balance and construction generally, will require extra care, resulting in a more expensive machine; moreover, while such a machine may run smoothly for a considerable time, any looseness or play that may develop later, causing a slight imbalance, will immediately set up excessive vibrations.

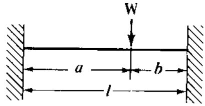
The formulas commonly used to determine critical speeds are sufficiently accurate for general purposes. There are cases, however, where the torque applied to a shaft has an important effect on its critical speed. Investigations have shown that the critical speeds of a uniform shaft are decreased as the applied torque is increased, and that there exist critical torques which will reduce the corresponding critical speed of the shaft to zero. A detailed analysis of the effects of applied torques on critical speeds may be found in a paper, "Critical Speeds of Uniform Shafts under Axial Torque," by Golumb and Rosenberg, presented at the First U.S. National Congress of Applied Mechanics in 1951.

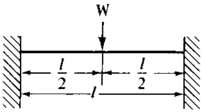
Formulas for Critical Speeds.—The critical speed formulas given in the accompanying table (from the paper on Critical Speed Calculation presented before the ASME by S. H. Weaver) apply to (1) shafts with single concentrated loads and (2) shafts carrying uniformly distributed loads. These formulas also cover different conditions as regards bearings. If the bearings are self-aligning or very short, the shaft is considered supported at the ends; whereas, if the bearings are long and rigid, the shaft is considered fixed. These formulas, for both concentrated and distributed loads, apply to vertical shafts as well as horizontal shafts, the critical speeds having the same value in both cases. The data required for the solution of critical speed problems are the same as for shaft deflection. As the shaft is usually of variable diameter and its stiffness is increased by a long hub, an ideal shaft of uniform diameter and equal stiffness must be assumed.

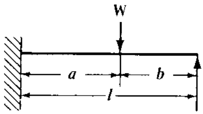
Critical Speed Formulas

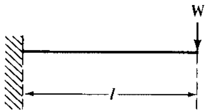
<table>
<tr><td colspan="3">Formulas for Single Concentrated Load</td></tr>
<tr>
<td>

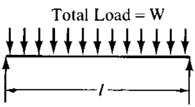
$N = 387,000 \frac{d^2}{ab}\sqrt{\frac{l}{W}}$
Bearings supported</td>
<td>

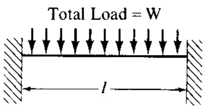
$N = 1,550,500 \frac{d^2}{l\sqrt{Wl}}$
Bearings supported</td>
<td>

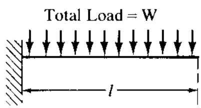
$N = 387,000 \frac{d^2 l}{ab}\sqrt{\frac{l}{Wab}}$
Bearings fixed</td>
</tr>
<tr>
<td>

$N = 3,100,850 \frac{d^2}{l\sqrt{Wl}}$
Bearings fixed</td>
<td>

$N = 775,200 \frac{d^2 l}{ab}\sqrt{\frac{l}{Wa(3l+b)}}$
One-fixed — One supported</td>
<td>

$N = 387,000 \frac{d^2}{l\sqrt{Wl}}$
One fixed — One free end</td>
</tr>
<tr><td colspan="3">Formulas for Distributed Loads—First Critical Speed</td></tr>
<tr>
<td>

$N = 2,232,500 \frac{d^2}{l\sqrt{Wl}}$
$N_1 = 4,760,000 \frac{d}{l^2}$
Bearings supported</td>
<td>

$N = 4,979,250 \frac{d^2}{l\sqrt{Wl}}$
$N_1 = 10,616,740 \frac{d}{l^2}$
Bearings fixed</td>
<td>

$N = 795,200 \frac{d^2}{l\sqrt{Wl}}$
$N_1 = 1,695,500 \frac{d}{l^2}$
One fixed—One free end</td>
</tr>
</table>

N = critical speed, RPM

N_1 = critical speed of shaft alone

d = diameter of shaft, in inches

W = load applied to shaft, in pounds

l = distance between centers of bearings, in inches

a and b = distances from bearings to load

In calculating critical speeds, the weight of the shaft is either neglected or, say, one-half to two-thirds of the weight is added to the concentrated load. The formulas apply to steel shafts having a modulus of elasticity E = 29,000,000. Although a shaft carrying a number of loads or a distributed load may have an infinite number of critical speeds, ordinarily it is the first critical speed that is of importance in engineering work. The first critical speed is obtained by the formulas given in the distributed loads portion of the table *Critical Speed Formulas*.

Angular Velocity

Angular Velocity of Rotating Bodies.—The angular velocity of a rotating body is the angle through which the body turns in a unit of time. Angular velocity is commonly expressed in terms of revolutions per minute, but in certain engineering applications it is necessary to express it as radians per second. By definition there are 2π radians in 360 degrees, or one revolution, so that one radian = $360 \quad 2\pi$ = 57.3 degrees. To convert angular velocity in revolutions per minute, n, to angular velocity in radians per second, ω, multiply by π and divide by 30:

$$\omega = \frac{\pi n}{30} \qquad (1)$$

The following table may be used to obtain angular velocity in radians per second for all numbers of revolutions per minute from 1 to 239.

Example: To find the angular velocity in radians per second of a flywheel making 97 revolutions per minute, locate 90 in the left-hand column and 7 at the top of the columns; at the intersection of the two lines, the angular velocity is read off as equal to 10.16 radians per second.

Linear Velocity of Points on a Rotating Body.—The linear velocity, v, of any point on a rotating body expressed in feet per second may be found by multiplying the angular velocity of the body in radians per second, ω, by the radius, r, in feet from the center of rotation to the point:

$$v = \omega r \qquad (2)$$

The metric SI units are v = meters per second; ω = radians per second, r = meters.

Angular Velocity in Revolutions per Minute Converted to Radians per Second

R.P.M.	Angular Velocity in Radians per Second									
	0	1	2	3	4	5	6	7	8	9
0	0.00	0.10	0.21	0.31	0.42	0.52	0.63	0.73	0.84	0.94
10	1.05	1.15	1.26	1.36	1.47	1.57	1.67	1.78	1.88	1.99
20	2.09	2.20	2.30	2.41	2.51	2.62	2.72	2.83	2.93	3.04
30	3.14	3.25	3.35	3.46	3.56	3.66	3.77	3.87	3.98	4.08
40	4.19	4.29	4.40	4.50	4.61	4.71	4.82	4.92	5.03	5.13
50	5.24	5.34	5.44	5.55	5.65	5.76	5.86	5.97	6.07	6.18
60	6.28	6.39	6.49	6.60	6.70	6.81	6.91	7.02	7.12	7.23
70	7.33	7.43	7.54	7.64	7.75	7.85	7.96	8.06	8.17	8.27
80	8.38	8.48	8.59	8.69	8.80	8.90	9.01	9.11	9.21	9.32
90	9.42	9.53	9.63	9.74	9.84	9.95	10.05	10.16	10.26	10.37
100	10.47	10.58	10.68	10.79	10.89	11.00	11.10	11.20	11.31	11.41
110	11.52	11.62	11.73	11.83	11.94	12.04	12.15	12.25	12.36	12.46
120	12.57	12.67	12.78	12.88	12.98	13.09	13.19	13.30	13.40	13.51
130	13.61	13.72	13.82	13.93	14.03	14.14	14.24	14.35	14.45	14.56
140	14.66	14.76	14.87	14.97	15.08	15.18	15.29	15.39	15.50	15.60
150	15.71	15.81	15.92	16.02	16.13	16.23	16.34	16.44	16.55	16.65
160	16.75	16.86	16.96	17.07	17.17	17.28	17.38	17.49	17.59	17.70
170	17.80	17.91	18.01	18.12	18.22	18.33	18.43	18.53	18.64	18.74
180	18.85	18.95	19.06	19.16	19.27	19.37	19.48	19.58	19.69	19.79
190	19.90	20.00	20.11	20.21	20.32	20.42	20.52	20.63	20.73	20.84
200	20.94	21.05	21.15	21.26	21.36	21.47	21.57	21.68	21.78	21.89
210	21.99	22.10	22.20	22.30	22.41	22.51	22.62	22.72	22.83	22.93
220	23.04	23.14	23.25	23.35	23.46	23.56	23.67	23.77	23.88	23.98
230	24.09	24.19	24.29	24.40	24.50	24.61	24.71	24.82	24.92	25.03

PENDULUMS

Types of Pendulums

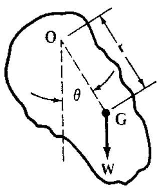
Types of Pendulums.—A *compound* or *physical* pendulum consists of any rigid body suspended from a fixed horizontal axis about which the body may oscillate in a vertical plane due to the action of gravity.

A *simple* or *mathematical* pendulum is similar to a compound pendulum except that the mass of the body is concentrated at a single point which is suspended from a fixed horizontal axis by a weightless cord. Actually, a simple pendulum cannot be constructed since it is impossible to have either a weightless cord or a body whose mass is entirely concentrated at one point. A good approximation, however, consists of a small, heavy bob suspended by a light, fine wire. If these conditions are not met by the pendulum, it should be considered as a compound pendulum.

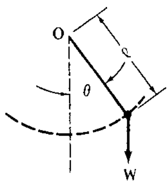
A *conical* pendulum is similar to a simple pendulum except that the weight suspended by the cord moves at a uniform speed around the circumference of a circle in a horizontal plane instead of oscillating back and forth in a vertical plane. The principle of the conical pendulum is employed in the Watt fly-ball governor.

A *torsional* pendulum in its simplest form consists of a disk fixed to a slender rod, the other end of which is fastened to a fixed frame. When the disc is twisted through some angle and released, it will then oscillate back and forth about the axis of the rod because of the torque exerted by the rod.

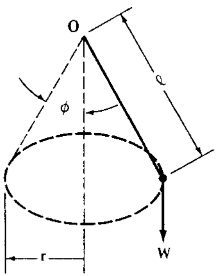
Four Types of Pendulum



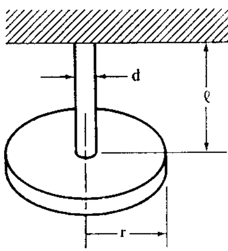
Physical Pendulum



Simple Pendulum



Conical Pendulum



Torsional Pendulum

W = Weight of Disk

Pendulum Calculations

Pendulum Formulas.—From the formulas that follow, the period of vibration or time required for one complete cycle back and forth may be determined for the types of pendulums shown in the accompanying diagram.

For a *simple* pendulum,

$$T = 2\pi\sqrt{\frac{l}{g}} \qquad (1)$$

where T = period in seconds for one complete cycle; g = acceleration due to gravity = 32.17 feet per second per second (approximately); and l is the length of the pendulum in feet as shown on the accompanying diagram.

For a *physical* or *compound* pendulum,

$$T = 2\pi\sqrt{\frac{k_o^2}{gr}} \qquad (2)$$

where k_0 = radius of gyration of the pendulum about the axis of rotation, in feet, and r is the distance from the axis of rotation to the center of gravity, in feet.

The metric SI units that can be used in the two above formulas are T = seconds; g = approximately 9.81 meters per second squared, which is the value for acceleration due to gravity; l = the length of the pendulum in meters; k_0 = the radius of gyration in meters, and r = the distance from the axis of rotation to the center of gravity, in meters.

Formulas (1) and (2) are accurate when the angle of oscillation θ shown in the diagram is very small. For θ equal to 22 degrees, these formulas give results that are too small by 1 per cent; for θ equal to 32 degrees, by 2 per cent.

For a *conical* pendulum, the time in seconds for one revolution is:

$$T = 2\pi\sqrt{\frac{l\cos\phi}{g}} \quad (3a) \qquad \text{or} \qquad T = 2\pi\sqrt{\frac{r\cot\phi}{g}} \quad (3b)$$

For a *torsional* pendulum consisting of a thin rod and a disk as shown in the figure

$$T = \frac{2}{3}\sqrt{\frac{\pi W r^2 l}{g d^4 G}} \qquad (4)$$

where W = weight of disk in pounds; r = radius of disk in feet; l = length of rod in feet; d = diameter of rod in feet; and G = modulus of elasticity in shear of the rod material in pounds per square inch.

The formula using metric SI units is:

$$\boldsymbol{T = 8\sqrt{\frac{\pi M r^2 l}{d^4 G}}}$$

where T = time in seconds for one complete oscillation; M = mass in kilograms; r = radius in meters; l = length of rod in meters; d = diameter of rod in meters; G = modulus of elasticity in shear of the rod material in pascals (newtons per meter squared). The same formula can be applied using millimeters, providing dimensions are expressed in millimeters throughout, and the modulus of elasticity in megapascals (newtons per millimeter squared).

FRICTION

Properties of Friction

Friction is the resistance to motion that takes place when one body is moved upon another, and is generally defined as "that force which acts between two bodies at their surface of contact, so as to resist their sliding on each other." According to the conditions under which sliding occurs, the force of friction, F, bears a certain relation to the force between the two bodies called the normal force N. The relation between force of friction and normal force is given by the *coefficient of friction*, generally denoted by the Greek letter μ. Thus:

$$F = \mu \times N \qquad \text{and} \qquad \mu = \frac{F}{N}$$

A body weighing 28 pounds rests on a horizontal surface. The force required to keep it in motion along the surface is 7 pounds. Find the coefficient of friction.

$$\mu = \frac{F}{N} = \frac{7}{28} = 0.25$$

If a body is placed on an inclined plane, the friction between the body and the plane will prevent it from sliding down the inclined surface, provided the angle of the plane with the horizontal is not too great. There will be a certain angle, however, at which the body will just barely be able to remain stationary, the frictional resistance being very nearly overcome by the tendency of the body to slide down. This angle is termed the angle of repose, and the tangent of this angle equals the coefficient of friction. The angle of repose is frequently denoted by the Greek letter θ. Thus, $\mu = \tan\theta$.

A greater force is required to start a body moving from a state of rest than to merely keep it in motion, because the *friction of rest* is greater than the *friction of motion*.

Laws of Friction.—The laws of friction for unlubricated or dry surfaces are summarized in the following statements.

1) For low pressures (normal force per unit area) the friction is directly proportional to the normal force between the two surfaces. As the pressure increases, the friction does not rise proportionally; but when the pressure becomes abnormally high, the friction increases at a rapid rate until seizing takes place.

2) The friction both in its total amount and its coefficient is independent of the areas in contact, so long as the normal force remains the same. This is true for moderate pressures only. For high pressures, this law is modified in the same way as in the first case.

3) At very low velocities the friction is independent of the velocity of rubbing. As the velocities increase, the friction decreases.

Lubricated Surfaces: For well lubricated surfaces, the laws of friction are considerably different from those governing dry or poorly lubricated surfaces.

1) The frictional resistance is almost independent of the pressure (normal force per unit area) if the surfaces are flooded with oil.

2) The friction varies directly as the speed, at low pressures; but for high pressures the friction is very great at low velocities, approaching a minimum at about two feet per second linear velocity, and afterwards increasing approximately as the square root of the speed.

3) For well lubricated surfaces the frictional resistance depends, to a very great extent, on the temperature, partly because of the change in the viscosity of the oil and partly because, for a journal bearing, the diameter of the bearing increases with the rise of temperature more rapidly than the diameter of the shaft, thus relieving the bearing of side pressure.

4) If the bearing surfaces are flooded with oil, the friction is almost independent of the nature of the material of the surfaces in contact. As the lubrication becomes less ample, the coefficient of friction becomes more dependent upon the material of the surfaces.

Influence of Friction on the Efficiency of Small Machine Elements.—Friction between machine parts lowers the efficiency of a machine. Average values of the efficiency, in per cent, of the most common machine elements when carefully made are ordinary bearings, 95 to 98; roller bearings, 98; ball bearings, 99; spur gears with cut teeth, including bearings, 99; bevel gears with cut teeth, including bearings, 98; belting, from 96 to 98; high-class silent power transmission chain, 97 to 99; roller chains, 95 to 97.

Coefficients of Friction.—Tables 1 and 2 provide representative values of static friction for various combinations of materials with dry (clean, unlubricated) and lubricated surfaces. The values for static or breakaway friction shown in these tables will generally be higher than the subsequent or sliding friction. Typically, the steel-on-steel static coefficient of 0.8 unlubricated will drop to 0.4 when sliding has been initiated; with oil lubrication, the value will drop from 0.16 to 0.03.

Many factors affect friction, and even slight deviations from normal or test conditions can produce wide variations. Accordingly, when using friction coefficients in design calculations, due allowance or factors of safety should be considered, and in critical applications, specific tests conducted to provide specific coefficients for material, geometry, and/or lubricant combinations.

Rolling Friction.—When a body rolls on a surface, the force resisting the motion is termed *rolling friction* or *rolling resistance*. Let W = total weight of rolling body or load on wheel, in pounds; r = radius of wheel, in inches; f = coefficient of rolling resistance, in inches. Then: resistance to rolling, in pounds = $(W \times f) \div r$.

The coefficient of rolling resistance varies with the conditions. For wood on wood it may be assumed as 0.06 inch; for iron on iron, 0.02 inch; iron on granite, 0.085 inch; iron on asphalt, 0.15 inch; and iron on wood, 0.22 inch.

The coefficient of rolling resistance, f, is in inches and is not the same as the sliding or static coefficient of friction given in Tables 1 and 2, which is a dimensionless ratio between frictional resistance and normal load. Various investigators are not in close agreement on the true values for these coefficients and the foregoing values should only be used for the approximate calculation of rolling resistance.

Table 1. Coefficients of Static Friction for Steel on Various Materials

Material	Coefficient of Friction, μ	
	Clean	Lubricated
Steel	0.8	0.16
Copper-lead alloy	0.22	…
Phosphor-bronze	0.35	…
Aluminum-bronze	0.45	…
Brass	0.35	0.19
Cast iron	0.4	0.21
Bronze	…	0.16
Sintered bronze	…	0.13
Hard carbon	0.14	0.11–0.14
Graphite	0.1	0.1
Tungsten carbide	0.4–0.6	0.1–0.2
Plexiglas	0.4–0.5	0.4–0.5
Polystyrene	0.3–0.35	0.3–0.35
Polythene	0.2	0.2
Teflon	0.04	0.04

Tables 1 and 2 used with permission from *The Friction and Lubrication of Solids*, Vol. 1, by Bowden and Tabor, Clarendon Press, Oxford, 1950.

Table 2. Coefficients of Static Friction for Various Materials Combinations

Material Combination	Coefficient of Friction, μ	
	Clean	Lubricated
Aluminum-aluminum	1.35	0.30
Cadmium-cadmium	0.5	0.05
Chromium-chromium	0.41	0.34
Copper-copper	1.0	0.08
Iron-iron	1.0	0.15–0.20
Magnesium-magnesium	0.6	0.08
Nickel-nickel	0.7	0.28
Platinum-platinum	1.2	0.25
Silver-silver	1.4	0.55
Zinc-zinc	0.6	0.04
Glass-glass	0.9–1.0	0.1–0.6
Glass-metal	0.5–0.7	0.2–0.3
Diamond-diamond	0.1	0.05–0.1
Diamond-metal	0.1–0.15	0.1
Sapphire-sapphire	0.2	0.2
Hard carbon on carbon	0.16	0.12–0.14
Graphite-graphite (in vacuum)	0.5–0.8	…
Graphite-graphite	0.1	0.1
Tungsten carbide-tungsten carbide	0.2–0.25	0.12
Plexiglas-plexiglas	0.8	0.8
Polystyrene-polystyrene	0.5	0.5
Teflon-Teflon	0.04	0.04
Nylon-nylon	0.15–0.25	…
Solids on rubber	1–4	…
Wood on wood (clean)	0.25–0.5	…
Wood on wood (wet)	0.2	…
Wood on metals (clean)	0.2–0.6	…
Wood on metals (wet)	0.2	…
Brick on wood	0.6	…
Leather on wood	0.3–0.4	…
Leather on metal (clean)	0.6	…
Leather on metal (wet)	0.4	…
Leather on metal (greasy)	0.2	…
Brake material on cast iron	0.4	…
Brake material on cast iron (wet)	0.2	…