TABLE OF CONTENTS

MATHEMATICS

NUMBERS, FRACTIONS, AND DECIMALS

ALGEBRA AND EQUATIONS

GEOMETRY

SOLUTION OF TRIANGLES

LOGARITHMS

NUMBERS, FRACTIONS, AND DECIMALS

Mathematical Signs and Commonly Used Abbreviations

$+$	Plus (sign of addition)	π	Pi (3.1416)
$+$	Positive	Σ	Sigma (sign of summation)
$-$	Minus (sign of subtraction)	ω	Omega (angles measured in radians)
$-$	Negative	g	Acceleration due to gravity (32.16 ft. per sec. per sec.)
$\pm\ (\mp)$	Plus or minus (minus or plus)	i (or j)	Imaginary quantity $(\sqrt{-1})$
$\times$	Multiplied by (multiplication sign)	sin	Sine
$\cdot$	Multiplied by (multiplication sign)	cos	Cosine
$\div$	Divided by (division sign)	tan	Tangent
$/$	Divided by (division sign)	cot	Cotangent
$:$	Is to (in proportion)	sec	Secant
$=$	Equals	csc	Cosecant
$\neq$	Is not equal to	vers	Versed sine
$\equiv$	Is identical to	covers	Coversed sine
$::$	Equals (in proportion)	$\sin^{-1} a$ $\arcsin a$	Arc the sine of which is a
$\cong$	Approximately equals	$(\sin a)^{-1}$	Reciprocal of sin a ($1 \div \sin a$)
$\approx$		$\sin^n x$	nth power of sin x
$>$	Greater than	$\sinh x$	Hyperbolic sine of x
$<$	Less than	$\cosh x$	Hyperbolic cosine of x
$\geq$	Greater than or equal to	Δ	Delta (increment of)
$\leq$	Less than or equal to	δ	Delta (variation of)
$\rightarrow$	Approaches as a limit	d	Differential (in calculus)
$\propto$	Varies directly as	∂	Partial differentiation (in calculus)
$\therefore$	Therefore	$\int$	Integral (in calculus)
$\sqrt{\ }$	Square root	$\int_b^a$	Integral between the limits a and b
$\sqrt[3]{\ }$	Cube root	!	$5! = 1 \times 2 \times 3 \times 4 \times 5$ (Factorial)
$\sqrt[4]{\ }$	4th root	$\angle$	Angle
$\sqrt[n]{\ }$	nth root½	∟	Right angle
a^2	a squared (2nd power of a)	$\perp$	Perpendicular to
a^3	a cubed (3rd power of a)	$\triangle$	Triangle
a^4	4th power of a	○	Circle
a^n	nth power of a	▱	Parallelogram
a^{-n}	$1 \div a^n$	$^\circ$	Degree (circular arc or temperature)
$\frac{1}{n}$	Reciprocal value of n	$'$	Minutes or feet
log	Logarithm	$''$	Seconds or inches
$\log_e$	Natural or Napierian logarithm	a'	a prime
ln		a''	a double prime
e	Base of natural logarithms (2.71828)	a_1	a sub one
lim	Limit value (of an expression)	a_2	a sub two
∞	Infinity	a_n	a sub n
α	Alpha	( )	Parentheses
β	Beta — commonly used to	[ ]	Brackets
γ	Gamma — denote angles	{ }	Braces
θ	Theta		
ϕ	Phi		
μ	Mu (coefficient of friction)		

Prime Numbers and Factors of Numbers

The *factors* of a given number are those numbers which when multiplied together give a product equal to that number; thus, 2 and 3 are factors of 6; and 5 and 7 are factors of 35.

A *prime number* is one which has no factors except itself and 1. Thus, 3, 5, 7, 11, etc., are prime numbers. A factor which is a prime number is called a *prime factor*.

The accompanying "Prime Number and Factor Tables" give the smallest prime factor of all odd numbers from 1 to 9600, and can be used for finding all the factors for numbers up to this limit. For example, find the factors of 931. In the column headed "900" and in the line indicated by "31" in the left-hand column, the smallest prime factor is found to be 7. As this leaves another factor 133 (since 931 ÷ 7 = 133), find the smallest prime factor of this number. In the column headed "100" and in the line "33", this is found to be 7, leaving a factor 19. This latter is a prime number; hence, the factors of 931 are $7 \times 7 \times 19$. Where no factor is given for a number in the factor table, it indicates that the number is a prime number.

The last page of the tables lists all prime numbers from 9551 through 18691; and can be used to identify quickly all unfactorable numbers in that range.

For factoring, the following general rules will be found useful:

2 is a factor of any number the right-hand figure of which is an even number or 0. Thus, $28 = 2 \times 14$, and $210 = 2 \times 105$.

3 is a factor of any number the sum of the figures of which is evenly divisible by 3. Thus, 3 is a factor of 1869, because $1 + 8 + 6 + 9 = 24 \div 3 = 8$.

4 is a factor of any number the two right-hand figures of which, considered as one number, are evenly divisible by 4. Thus, 1844 has a factor 4, because $44 \div 4 = 11$.

5 is a factor of any number the right-hand figure of which is 0 or 5. Thus, $85 = 5 \times 17$; $70 = 5 \times 14$.

Tables of prime numbers and factors of numbers are particularly useful for calculations involving change-gear ratios for compound gearing, dividing heads, gear-generating machines, and mechanical designs having gear trains.

Example 1: A set of four gears is required in a mechanical design to provide an overall gear ratio of $4104 \div 1200$. Furthermore, no gear in the set is to have more than 120 teeth or less than 24 teeth. Determine the tooth numbers.

First, as explained previously, the factors of 4104 are determined to be: $2 \times 2 \times 2 \times 3 \times 3 \times 57 = 4104$. Next, the factors of 1200 are determined: $2 \times 2 \times 2 \times 2 \times 5 \times 5 \times 3 = 1200$. Therefore $\frac{4104}{1200} = \frac{2 \times 2 \times 2 \times 3 \times 3 \times 57}{2 \times 2 \times 2 \times 2 \times 5 \times 5 \times 3} = \frac{72 \times 57}{24 \times 50}$. If the factors had been combined differently, say, to give $\frac{72 \times 57}{16 \times 75}$, then the 16-tooth gear in the denominator would not satisfy the requirement of no less than 24 teeth.

Example 2: Factor the number 25078 into two numbers neither of which is larger than 200.

The first factor of 25078 is obviously 2, leaving $25078 \div 2 = 12539$ to be factored further. However, from the last table, *Prime Numbers from 9551 to 18691*, it is seen that 12539 is a prime number; therefore, no solution exists.

Prime Number and Factor Table for 1 to 1199

From To	0 100	100 200	200 300	300 400	400 500	500 600	600 700	700 800	800 900	900 1000	1000 1100	1100 1200
1	P	P	3	7	P	3	P	P	3	17	7	3
3	P	P	7	3	13	P	3	19	11	3	17	P
5	P	3	5	5	3	5	5	3	5	5	3	5
7	P	P	3	P	11	3	P	7	3	P	19	3
9	3	P	11	3	P	P	3	P	P	3	P	P
11	P	3	P	P	3	7	13	3	P	P	3	11
13	P	P	3	P	7	3	P	23	3	11	P	3
15	3	5	5	3	5	5	3	5	5	3	5	5
17	P	3	7	P	3	11	P	3	19	7	3	P
19	P	7	3	11	P	3	P	P	3	P	P	3
21	3	11	13	3	P	P	3	7	P	3	P	19
23	P	3	P	17	3	P	7	3	P	13	3	P
25	5	5	3	5	5	3	5	5	3	5	5	3
27	3	P	P	3	7	17	3	P	P	3	13	7
29	P	3	P	7	3	23	17	3	P	P	3	P
31	P	P	3	P	P	3	P	17	3	7	P	3
33	3	7	P	3	P	13	3	P	7	3	P	11
35	5	3	5	5	3	5	5	3	5	5	3	5
37	P	P	3	P	19	3	7	11	3	P	17	3
39	3	P	P	3	P	7	3	P	P	3	P	17
41	P	3	P	11	3	P	P	3	29	P	3	7
43	P	11	3	7	P	3	P	P	3	23	7	3
45	3	5	5	3	5	5	3	5	5	3	5	5
47	P	3	13	P	3	P	P	3	7	P	3	31
49	7	P	3	P	P	3	11	7	3	13	P	3
51	3	P	P	3	11	19	3	P	23	3	P	P
53	P	3	11	P	3	7	P	3	P	P	3	P
55	5	5	3	5	5	3	5	5	3	5	5	3
57	3	P	P	3	P	P	3	P	P	3	7	13
59	P	3	7	P	3	13	P	3	P	7	3	19
61	P	7	3	19	P	3	P	P	3	31	P	3
63	3	P	P	3	P	P	3	7	P	3	P	P
65	5	3	5	5	3	5	5	3	5	5	3	5
67	P	P	3	P	P	3	23	13	3	P	11	3
69	3	13	P	3	7	P	3	P	11	3	P	7
71	P	3	P	7	3	P	11	3	13	P	3	P
73	P	P	3	P	11	3	P	P	3	7	29	3
75	3	5	5	3	5	5	3	5	5	3	5	5
77	7	3	P	13	3	P	P	3	P	P	3	11
79	P	P	3	P	P	3	7	19	3	11	13	3
81	3	P	P	3	13	7	3	11	P	3	23	P
83	P	3	P	P	3	11	P	3	P	P	3	7
85	5	5	3	5	5	3	5	5	3	5	5	3
87	3	11	7	3	P	P	3	P	P	3	P	P
89	P	3	17	P	3	19	13	3	7	23	3	29
91	7	P	3	17	P	3	P	7	3	P	P	3
93	3	P	P	3	17	P	3	13	19	3	P	P
95	5	3	5	5	3	5	5	3	5	5	3	5
97	P	P	3	P	7	3	17	P	3	P	P	3
99	3	P	13	3	P	P	3	17	29	3	7	11

Prime Number and Factor Table for 1201 to 2399

From To	1200 1300	1300 1400	1400 1500	1500 1600	1600 1700	1700 1800	1800 1900	1900 2000	2000 2100	2100 2200	2200 2300	2300 2400
1	P	P	3	19	P	3	P	P	3	11	31	3
3	3	P	23	3	7	13	3	11	P	3	P	7
5	5	3	5	5	3	5	5	3	5	5	3	5
7	17	P	3	11	P	3	13	P	3	7	P	3
9	3	7	P	3	P	P	3	23	7	3	47	P
11	7	3	17	P	3	29	P	3	P	P	3	P
13	P	13	3	17	P	3	7	P	3	P	P	3
15	3	5	5	3	5	5	3	5	5	3	5	5
17	P	3	13	37	3	17	23	3	P	29	3	7
19	23	P	3	7	P	3	17	19	3	13	7	3
21	3	P	7	3	P	P	3	17	43	3	P	11
23	P	3	P	P	3	P	P	3	7	11	3	23
25	5	5	3	5	5	3	5	5	3	5	5	3
27	3	P	P	3	P	11	3	41	P	3	17	13
29	P	3	P	11	3	7	31	3	P	P	3	17
31	P	11	3	P	7	3	P	P	3	P	23	3
33	3	31	P	3	23	P	3	P	19	3	7	P
35	5	3	5	5	3	5	5	3	5	5	3	5
37	P	7	3	29	P	3	11	13	3	P	P	3
39	3	13	P	3	11	37	3	7	P	3	P	P
41	17	3	11	23	3	P	7	3	13	P	3	P
43	11	17	3	P	31	3	19	29	3	P	P	3
45	3	5	5	3	5	5	3	5	5	3	5	5
47	29	3	P	7	3	P	P	3	23	19	3	P
49	P	19	3	P	17	3	43	P	3	7	13	3
51	3	7	P	3	13	17	3	P	7	3	P	P
53	7	3	P	P	3	P	17	3	P	P	3	13
55	5	5	3	5	5	3	5	5	3	5	5	3
57	3	23	31	3	P	7	3	19	11	3	37	P
59	P	3	P	P	3	P	11	3	29	17	3	7
61	13	P	3	7	11	3	P	37	3	P	7	3
63	3	29	7	3	P	41	3	13	P	3	31	17
65	5	3	5	5	3	5	5	3	5	5	3	5
67	7	P	3	P	P	3	P	7	3	11	P	3
69	3	37	13	3	P	29	3	11	P	3	P	23
71	31	3	P	P	3	7	P	3	19	13	3	P
73	19	P	3	11	7	3	P	P	3	41	P	3
75	3	5	5	3	5	5	3	5	5	3	5	5
77	P	3	7	19	3	P	P	3	31	7	3	P
79	P	7	3	P	23	3	P	P	3	P	43	3
81	3	P	P	3	41	13	3	7	P	3	P	P
83	P	3	P	P	3	P	7	3	P	37	3	P
85	5	5	3	5	5	3	5	5	3	5	5	3
87	3	19	P	3	7	P	3	P	P	3	P	7
89	P	3	P	7	3	P	P	3	P	11	3	P
91	P	13	3	37	19	3	31	11	3	7	29	3
93	3	7	P	3	P	11	3	P	7	3	P	P
95	5	3	5	5	3	5	5	3	5	5	3	5
97	P	11	3	P	P	3	7	P	3	13	P	3
99	3	P	P	3	P	7	3	P	P	3	11	P

Prime Number and Factor Table for 2401 to 3599

From To	2400 2500	2500 2600	2600 2700	2700 2800	2800 2900	2900 3000	3000 3100	3100 3200	3200 3300	3300 3400	3400 3500	3500 3600
1	7	41	3	37	P	3	P	7	3	P	19	3
3	3	P	19	3	P	P	3	29	P	3	41	31
5	5	3	5	5	3	5	5	3	5	5	3	5
7	29	23	3	P	7	3	31	13	3	P	P	3
9	3	13	P	3	53	P	3	P	P	3	7	11
11	P	3	7	P	3	41	P	3	13	7	3	P
13	19	7	3	P	29	3	23	11	3	P	P	3
15	3	5	5	3	5	5	3	5	5	3	5	5
17	P	3	P	11	3	P	7	3	P	31	3	P
19	41	11	3	P	P	3	P	P	3	P	13	3
21	3	P	P	3	7	23	3	P	P	3	11	7
23	P	3	43	7	3	37	P	3	11	P	3	13
25	5	5	3	5	5	3	5	5	3	5	5	3
27	3	7	37	3	11	P	3	53	7	3	23	P
29	7	3	11	P	3	29	13	3	P	P	3	P
31	11	P	3	P	19	3	7	31	3	P	47	3
33	3	17	P	3	P	7	3	13	53	3	P	P
35	5	3	5	5	3	5	5	3	5	5	3	5
37	P	43	3	7	P	3	P	P	3	47	7	3
39	3	P	7	3	17	P	3	43	41	3	19	P
41	P	3	19	P	3	17	P	3	7	13	3	P
43	7	P	3	13	P	3	17	7	3	P	11	3
45	3	5	5	3	5	5	3	5	5	3	5	5
47	P	3	P	41	3	7	11	3	17	P	3	P
49	31	P	3	P	7	3	P	47	3	17	P	3
51	3	P	11	3	P	13	3	23	P	3	7	53
53	11	3	7	P	3	P	43	3	P	7	3	11
55	5	5	3	5	5	3	5	5	3	5	5	3
57	3	P	P	3	P	P	3	7	P	3	P	P
59	P	3	P	31	3	11	7	3	P	P	3	P
61	23	13	3	11	P	3	P	29	3	P	P	3
63	3	11	P	3	7	P	3	P	13	3	P	7
65	5	3	5	5	3	5	5	3	5	5	3	5
67	P	17	3	P	47	3	P	P	3	7	P	3
69	3	7	17	3	19	P	3	P	7	3	P	43
71	7	3	P	17	3	P	37	3	P	P	3	P
73	P	31	3	47	13	3	7	19	3	P	23	3
75	3	5	5	3	5	5	3	5	5	3	5	5
77	P	3	P	P	3	13	17	3	29	11	3	7
79	37	P	3	7	P	3	P	11	3	31	7	3
81	3	29	7	3	43	11	3	P	17	3	59	P
83	13	3	P	11	3	19	P	3	7	17	3	P
85	5	5	3	5	5	3	5	5	3	5	5	3
87	3	13	P	3	P	29	3	P	19	3	11	17
89	19	3	P	P	3	7	P	3	11	P	3	37
91	47	P	3	P	7	3	11	P	3	P	P	3
93	3	P	P	3	11	41	3	31	37	3	7	P
95	5	3	5	5	3	5	5	3	5	5	3	5
97	11	7	3	P	P	3	19	23	3	43	13	3
99	3	23	P	3	13	P	3	7	P	3	P	59

Prime Number and Factor Table for 3601 to 4799

From To	3600 3700	3700 3800	3800 3900	3900 4000	4000 4100	4100 4200	4200 4300	4300 4400	4400 4500	4500 4600	4600 4700	4700 4800
1	13	P	3	47	P	3	P	11	3	7	43	3
3	3	7	P	3	P	11	3	13	7	3	P	P
5	5	3	5	5	3	5	5	3	5	5	3	5
7	P	11	3	P	P	3	7	59	3	P	17	3
9	3	P	13	3	19	7	3	31	P	3	11	17
11	23	3	37	P	3	P	P	3	11	13	3	7
13	P	47	3	7	P	3	11	19	3	P	7	3
15	3	5	5	3	5	5	3	5	5	3	5	5
17	P	3	11	P	3	23	P	3	7	P	3	53
19	7	P	3	P	P	3	P	7	3	P	31	3
21	3	61	P	3	P	13	3	29	P	3	P	P
23	P	3	P	P	3	7	41	3	P	P	3	P
25	5	5	3	5	5	3	5	5	3	5	5	3
27	3	P	43	3	P	P	3	P	19	3	7	29
29	19	3	7	P	3	P	P	3	43	7	3	P
31	P	7	3	P	29	3	P	61	3	23	11	3
33	3	P	P	3	37	P	3	7	11	3	41	P
35	5	3	5	5	3	5	5	3	5	5	3	5
37	P	37	3	31	11	3	19	P	3	13	P	3
39	3	P	11	3	7	P	3	P	23	3	P	7
41	11	3	23	7	3	41	P	3	P	19	3	11
43	P	19	3	P	13	3	P	43	3	7	P	3
45	3	5	5	3	5	5	3	5	5	3	5	5
47	7	3	P	P	3	11	31	3	P	P	3	47
49	41	23	3	11	P	3	7	P	3	P	P	3
51	3	11	P	3	P	7	3	19	P	3	P	P
53	13	3	P	59	3	P	P	3	61	29	3	7
55	5	5	3	5	5	3	5	5	3	5	5	3
57	3	13	7	3	P	P	3	P	P	3	P	67
59	P	3	17	37	3	P	P	3	7	47	3	P
61	7	P	3	17	31	3	P	7	3	P	59	3
63	3	53	P	3	17	23	3	P	P	3	P	11
65	5	3	5	5	3	5	5	3	5	5	3	5
67	19	P	3	P	7	3	17	11	3	P	13	3
69	3	P	53	3	13	11	3	17	41	3	7	19
71	P	3	7	11	3	43	P	3	17	7	3	13
73	P	7	3	29	P	3	P	P	3	17	P	3
75	3	5	5	3	5	5	3	5	5	3	5	5
77	P	3	P	41	3	P	7	3	11	23	3	17
79	13	P	3	23	P	3	11	29	3	19	P	3
81	3	19	P	3	7	37	3	13	P	3	31	7
83	29	3	11	7	3	47	P	3	P	P	3	P
85	5	5	3	5	5	3	5	5	3	5	5	3
87	3	7	13	3	61	53	3	41	7	3	43	P
89	7	3	P	P	3	59	P	3	67	13	3	P
91	P	17	3	13	P	3	7	P	3	P	P	3
93	3	P	17	3	P	7	3	23	P	3	13	P
95	5	3	5	5	3	5	5	3	5	5	3	5
97	P	P	3	7	17	3	P	P	3	P	7	3
99	3	29	7	3	P	13	3	53	11	3	37	P

Prime Number and Factor Table for 4801 to 5999

From To	4800 4900	4900 5000	5000 5100	5100 5200	5200 5300	5300 5400	5400 5500	5500 5600	5600 5700	5700 5800	5800 5900	5900 6000
1	P	13	3	P	7	3	11	P	3	P	P	3
3	3	P	P	3	11	P	3	P	13	3	7	P
5	5	3	5	5	3	5	5	3	5	5	3	5
7	11	7	3	P	41	3	P	P	3	13	P	3
9	3	P	P	3	P	P	3	7	71	3	37	19
11	17	3	P	19	3	47	7	3	31	P	3	23
13	P	17	3	P	13	3	P	37	3	29	P	3
15	3	5	5	3	5	5	3	5	5	3	5	5
17	P	3	29	7	3	13	P	3	41	P	3	61
19	61	P	3	P	17	3	P	P	3	7	11	3
21	3	7	P	3	23	17	3	P	7	3	P	31
23	7	3	P	47	3	P	11	3	P	59	3	P
25	5	5	3	5	5	3	5	5	3	5	5	3
27	3	13	11	3	P	7	3	P	17	3	P	P
29	11	3	47	23	3	73	61	3	13	17	3	7
31	P	P	3	7	P	3	P	P	3	11	7	3
33	3	P	7	3	P	P	3	11	43	3	19	17
35	5	3	5	5	3	5	5	3	5	5	3	5
37	7	P	3	11	P	3	P	7	3	P	13	3
39	3	11	P	3	13	19	3	29	P	3	P	P
41	47	3	71	53	3	7	P	3	P	P	3	13
43	29	P	3	37	7	3	P	23	3	P	P	3
45	3	5	5	3	5	5	3	5	5	3	5	5
47	37	3	7	P	3	P	13	3	P	7	3	19
49	13	7	3	19	29	3	P	31	3	P	P	3
51	3	P	P	3	59	P	3	7	P	3	P	11
53	23	3	31	P	3	53	7	3	P	11	3	P
55	5	5	3	5	5	3	5	5	3	5	5	3
57	3	P	13	3	7	11	3	P	P	3	P	7
59	43	3	P	7	3	23	53	3	P	13	3	59
61	P	11	3	13	P	3	43	67	3	7	P	3
63	3	7	61	3	19	31	3	P	7	3	11	67
65	5	3	5	5	3	5	5	3	5	5	3	5
67	31	P	3	P	23	3	7	19	3	73	P	3
69	3	P	37	3	11	7	3	P	P	3	P	47
71	P	3	11	P	3	41	P	3	53	29	3	7
73	11	P	3	7	P	3	13	P	3	23	7	3
75	3	5	5	3	5	5	3	5	5	3	5	5
77	P	3	P	31	3	19	P	3	7	53	3	43
79	7	13	3	P	P	3	P	7	3	P	P	3
81	3	17	P	3	P	P	3	P	13	3	P	P
83	19	3	13	71	3	7	P	3	P	P	3	31
85	5	5	3	5	5	3	5	5	3	5	5	3
87	3	P	P	3	17	P	3	37	11	3	7	P
89	P	3	7	P	3	17	11	3	P	7	3	53
91	67	7	3	29	11	3	17	P	3	P	43	3
93	3	P	11	3	67	P	3	7	P	3	71	13
95	5	3	5	5	3	5	5	3	5	5	3	5
97	59	19	3	P	P	3	23	29	3	11	P	3
99	3	P	P	3	7	P	3	11	41	3	17	7

Prime Number and Factor Table for 6001 to 7199

From To	6000 6100	6100 6200	6200 6300	6300 6400	6400 6500	6500 6600	6600 6700	6700 6800	6800 6900	6900 7000	7000 7100	7100 7200
1	17	P	3	P	37	3	7	P	3	67	P	3
3	3	17	P	3	19	7	3	P	P	3	47	P
5	5	3	5	5	3	5	5	3	5	5	3	5
7	P	31	3	7	43	3	P	19	3	P	7	3
9	3	41	7	3	13	23	3	P	11	3	43	P
11	P	3	P	P	3	17	11	3	7	P	3	13
13	7	P	3	59	11	3	17	7	3	31	P	3
15	3	5	5	3	5	5	3	5	5	3	5	5
17	11	3	P	P	3	7	13	3	17	P	3	11
19	13	29	3	71	7	3	P	P	3	11	P	3
21	3	P	P	3	P	P	3	11	19	3	7	P
23	19	3	7	P	3	11	37	3	P	7	3	17
25	5	5	3	5	5	3	5	5	3	5	5	3
27	3	11	13	3	P	61	3	7	P	3	P	P
29	P	3	P	P	3	P	7	3	P	13	3	P
31	37	P	3	13	59	3	19	53	3	29	79	3
33	3	P	23	3	7	47	3	P	P	3	13	7
35	5	3	5	5	3	5	5	3	5	5	3	5
37	P	17	3	P	41	3	P	P	3	7	31	3
39	3	7	17	3	47	13	3	23	7	3	P	11
41	7	3	79	17	3	31	29	3	P	11	3	37
43	P	P	3	P	17	3	7	11	3	53	P	3
45	3	5	5	3	5	5	3	5	5	3	5	5
47	P	3	P	11	3	P	17	3	41	P	3	7
49	23	11	3	7	P	3	61	17	3	P	7	3
51	3	P	7	3	P	P	3	43	13	3	11	P
53	P	3	13	P	3	P	P	3	7	17	3	23
55	5	5	3	5	5	3	5	5	3	5	5	3
57	3	47	P	3	11	79	3	29	P	3	P	17
59	73	3	11	P	3	7	P	3	19	P	3	P
61	11	61	3	P	7	3	P	P	3	P	23	3
63	3	P	P	3	23	P	3	P	P	3	7	13
65	5	3	5	5	3	5	5	3	5	5	3	5
67	P	7	3	P	29	3	59	67	3	P	37	3
69	3	31	P	3	P	P	3	7	P	3	P	67
71	13	3	P	23	3	P	7	3	P	P	3	71
73	P	P	3	P	P	3	P	13	3	19	11	3
75	3	5	5	3	5	5	3	5	5	3	5	5
77	59	3	P	7	3	P	11	3	13	P	3	P
79	P	37	3	P	11	3	P	P	3	7	P	3
81	3	7	11	3	P	P	3	P	7	3	73	43
83	7	3	61	13	3	29	41	3	P	P	3	11
85	5	5	3	5	5	3	5	5	3	5	5	3
87	3	23	P	3	13	7	3	11	71	3	19	P
89	P	3	19	P	3	11	P	3	83	29	3	7
91	P	41	3	7	P	3	P	P	3	P	7	3
93	3	11	7	3	43	19	3	P	61	3	41	P
95	5	3	5	5	3	5	5	3	5	5	3	5
97	7	P	3	P	73	3	37	7	3	P	47	3
99	3	P	P	3	67	P	3	13	P	3	31	23

Prime Number and Factor Table for 7201 to 8399

From To	7200 7300	7300 7400	7400 7500	7500 7600	7600 7700	7700 7800	7800 7900	7900 8000	8000 8100	8100 8200	8200 8300	8300 8400
1	19	7	3	13	11	3	29	P	3	P	59	3
3	3	67	11	3	P	P	3	7	53	3	13	19
5	5	3	5	5	3	5	5	3	5	5	3	5
7	P	P	3	P	P	3	37	P	3	11	29	3
9	3	P	31	3	7	13	3	11	P	3	P	7
11	P	3	P	7	3	11	73	3	P	P	3	P
13	P	71	3	11	23	3	13	41	3	7	43	3
15	3	5	5	3	5	5	3	5	5	3	5	5
17	7	3	P	P	3	P	P	3	P	P	3	P
19	P	13	3	73	19	3	7	P	3	23	P	3
21	3	P	41	3	P	7	3	89	13	3	P	53
23	31	3	13	P	3	P	P	3	71	P	3	7
25	5	5	3	5	5	3	5	5	3	5	5	3
27	3	17	7	3	29	P	3	P	23	3	19	11
29	P	3	17	P	3	59	P	3	7	11	3	P
31	7	P	3	17	13	3	41	7	3	47	P	3
33	3	P	P	3	17	11	3	P	29	3	P	13
35	5	3	5	5	3	5	5	3	5	5	3	5
37	P	11	3	P	7	3	17	P	3	79	P	3
39	3	41	43	3	P	71	3	17	P	3	7	31
41	13	3	7	P	3	P	P	3	11	7	3	19
43	P	7	3	19	P	3	11	13	3	17	P	3
45	3	5	5	3	5	5	3	5	5	3	5	5
47	P	3	11	P	3	61	7	3	13	P	3	17
49	11	P	3	P	P	3	47	P	3	29	73	3
51	3	P	P	3	7	23	3	P	83	3	37	7
53	P	3	29	7	3	P	P	3	P	31	3	P
55	5	5	3	5	5	3	5	5	3	5	5	3
57	3	7	P	3	13	P	3	73	7	3	23	61
59	7	3	P	P	3	P	29	3	P	41	3	13
61	53	17	3	P	47	3	7	19	3	P	11	3
63	3	37	17	3	79	7	3	P	11	3	P	P
65	5	3	5	5	3	5	5	3	5	5	3	5
67	13	53	3	7	11	3	P	31	3	P	7	3
69	3	P	7	3	P	17	3	13	P	3	P	P
71	11	3	31	67	3	19	17	3	7	P	3	11
73	7	73	3	P	P	3	P	7	3	11	P	3
75	3	5	5	3	5	5	3	5	5	3	5	5
77	19	3	P	P	3	7	P	3	41	13	3	P
79	29	47	3	11	7	3	P	79	3	P	17	3
81	3	11	P	3	P	31	3	23	P	3	7	17
83	P	3	7	P	3	43	P	3	59	7	3	83
85	5	5	3	5	5	3	5	5	3	5	5	3
87	3	83	P	3	P	13	3	7	P	3	P	P
89	37	3	P	P	3	P	7	3	P	19	3	P
91	23	19	3	P	P	3	13	61	3	P	P	3
93	3	P	59	3	7	P	3	P	P	3	P	7
95	5	3	5	5	3	5	5	3	5	5	3	5
97	P	13	3	71	43	3	53	11	3	7	P	3
99	3	7	P	3	P	11	3	19	7	3	43	37

Prime Number and Factor Table for 8401 to 9599

From To	8400 8500	8500 8600	8600 8700	8700 8800	8800 8900	8900 9000	9000 9100	9100 9200	9200 9300	9300 9400	9400 9500	9500 9600
1	31	P	3	7	13	3	P	19	3	71	7	3
3	3	11	7	3	P	29	3	P	P	3	P	13
5	5	3	5	5	3	5	5	3	5	5	3	5
7	7	47	3	P	P	3	P	7	3	41	23	3
9	3	67	P	3	23	59	3	P	P	3	97	37
11	13	3	79	31	3	7	P	3	61	P	3	P
13	47	P	3	P	7	3	P	13	3	67	P	3
15	3	5	5	3	5	5	3	5	5	3	5	5
17	19	3	7	23	3	37	71	3	13	7	3	31
19	P	7	3	P	P	3	29	11	3	P	P	3
21	3	P	37	3	P	11	3	7	P	3	P	P
23	P	3	P	11	3	P	7	3	23	P	3	89
25	5	5	3	5	5	3	5	5	3	5	5	3
27	3	P	P	3	7	79	3	P	P	3	11	7
29	P	3	P	7	3	P	P	3	11	19	3	13
31	P	19	3	P	P	3	11	23	3	7	P	3
33	3	7	89	3	11	P	3	P	7	3	P	P
35	5	3	5	5	3	5	5	3	5	5	3	5
37	11	P	3	P	P	3	7	P	3	P	P	3
39	3	P	53	3	P	7	3	13	P	3	P	P
41	23	3	P	P	3	P	P	3	P	P	3	7
43	P	P	3	7	37	3	P	41	3	P	7	3
45	3	5	5	3	5	5	3	5	5	3	5	5
47	P	3	P	P	3	23	83	3	7	13	3	P
49	7	83	3	13	P	3	P	7	3	P	11	3
51	3	17	41	3	53	P	3	P	11	3	13	P
53	79	3	17	P	3	7	11	3	19	47	3	41
55	5	5	3	5	5	3	5	5	3	5	5	3
57	3	43	11	3	17	13	3	P	P	3	7	19
59	11	3	7	19	3	17	P	3	47	7	3	11
61	P	7	3	P	P	3	13	P	3	11	P	3
63	3	P	P	3	P	P	3	7	59	3	P	73
65	5	3	5	5	3	5	5	3	5	5	3	5
67	P	13	3	11	P	3	P	89	3	17	P	3
69	3	11	P	3	7	P	3	53	13	3	17	7
71	43	3	13	7	3	P	47	3	73	P	3	17
73	37	P	3	31	19	3	43	P	3	7	P	3
75	3	5	5	3	5	5	3	5	5	3	5	5
77	7	3	P	67	3	47	29	3	P	P	3	61
79	61	23	3	P	13	3	7	67	3	83	P	3
81	3	P	P	3	83	7	3	P	P	3	19	11
83	17	3	19	P	3	13	31	3	P	11	3	7
85	5	5	3	5	5	3	5	5	3	5	5	3
87	3	31	7	3	P	11	3	P	37	3	53	P
89	13	3	P	11	3	89	61	3	7	41	3	43
91	7	11	3	59	17	3	P	7	3	P	P	3
93	3	13	P	3	P	17	3	29	P	3	11	53
95	5	3	5	5	3	5	5	3	5	5	3	5
97	29	P	3	19	7	3	11	17	3	P	P	3
99	3	P	P	3	11	P	3	P	17	3	7	29

Prime Numbers from 9551 to 18691

9551	10181	10853	11497	12157	12763	13417	14071	14747	15361	16001	16693	17387	18043
9587	10193	10859	11503	12161	12781	13421	14081	14753	15373	16007	16699	17389	18047
9601	10211	10861	11519	12163	12791	13441	14083	14759	15377	16033	16703	17393	18049
9613	10223	10867	11527	12197	12799	13451	14087	14767	15383	16057	16729	17401	18059
9619	10243	10883	11549	12203	12809	13457	14107	14771	15391	16061	16741	17417	18061
9623	10247	10889	11551	12211	12821	13463	14143	14779	15401	16063	16747	17419	18077
9629	10253	10891	11579	12227	12823	13469	14149	14783	15413	16067	16759	17431	18089
9631	10259	10903	11587	12239	12829	13477	14153	14797	15427	16069	16763	17443	18097
9643	10267	10909	11593	12241	12841	13487	14159	14813	15439	16073	16787	17449	18119
9649	10271	10937	11597	12251	12853	13499	14173	14821	15443	16087	16811	17467	18121
9661	10273	10939	11617	12253	12889	13513	14177	14827	15451	16091	16823	17471	18127
9677	10289	10949	11621	12263	12893	13523	14197	14831	15461	16097	16829	17477	18131
9679	10301	10957	11633	12269	12899	13537	14207	14843	15467	16103	16831	17483	18133
9689	10303	10973	11657	12277	12907	13553	14221	14851	15473	16111	16843	17489	18143
9697	10313	10979	11677	12281	12911	13567	14243	14867	15493	16127	16871	17491	18149
9719	10321	10987	11681	12289	12917	13577	14249	14869	15497	16139	16879	17497	18169
9721	10331	10993	11689	12301	12919	13591	14251	14879	15511	16141	16883	17509	18181
9733	10333	11003	11699	12323	12923	13597	14281	14887	15527	16183	16889	17519	18191
9739	10337	11027	11701	12329	12941	13613	14293	14891	15541	16187	16901	17539	18199
9743	10343	11047	11717	12343	12953	13619	14303	14897	15551	16189	16903	17551	18211
9749	10357	11057	11719	12347	12959	13627	14321	14923	15559	16193	16921	17569	18217
9767	10369	11059	11731	12373	12967	13633	14323	14929	15569	16217	16927	17573	18223
9769	10391	11069	11743	12377	12973	13649	14327	14939	15581	16223	16931	17579	18229
9781	10399	11071	11777	12379	12979	13669	14341	14947	15583	16229	16937	17581	18233
9787	10427	11083	11779	12391	12983	13679	14347	14951	15601	16231	16943	17597	18251
9791	10429	11087	11783	12401	13001	13681	14369	14957	15607	16249	16963	17599	18253
9803	10433	11093	11789	12409	13003	13687	14387	14969	15619	16253	16979	17609	18257
9811	10453	11113	11801	12413	13007	13691	14389	14983	15629	16267	16981	17623	18269
9817	10457	11117	11807	12421	13009	13693	14401	15013	15641	16273	16987	17627	18287
9829	10459	11119	11813	12433	13033	13697	14407	15017	15643	16301	16993	17657	18289
9833	10463	11131	11821	12437	13037	13709	14411	15031	15647	16319	17011	17659	18301
9839	10477	11149	11827	12451	13043	13711	14419	15053	15649	16333	17021	17669	18307
9851	10487	11159	11831	12457	13049	13721	14423	15061	15661	16339	17027	17681	18311
9857	10499	11161	11833	12473	13063	13723	14431	15073	15667	16349	17029	17683	18313
9859	10501	11171	11839	12479	13093	13729	14437	15077	15671	16361	17033	17707	18329
9871	10513	11173	11863	12487	13099	13751	14447	15083	15679	16363	17041	17713	18341
9883	10529	11177	11867	12491	13103	13757	14449	15091	15683	16369	17047	17729	18353
9887	10531	11197	11887	12497	13109	13759	14461	15101	15727	16381	17053	17737	18367
9901	10559	11213	11897	12503	13121	13763	14479	15107	15731	16411	17077	17747	18371
9907	10567	11239	11903	12511	13127	13781	14489	15121	15733	16417	17093	17749	18379
9923	10589	11243	11909	12517	13147	13789	14503	15131	15737	16421	17099	17761	18397
9929	10597	11251	11923	12527	13151	13799	14519	15137	15739	16427	17107	17783	18401
9931	10601	11257	11927	12539	13159	13807	14533	15139	15749	16433	17117	17789	18413
9941	10607	11261	11933	12541	13163	13829	14537	15149	15761	16447	17123	17791	18427
9949	10613	11273	11939	12547	13171	13831	14543	15161	15767	16451	17137	17807	18433
9967	10627	11279	11941	12553	13177	13841	14549	15173	15773	16453	17159	17827	18439
9973	10631	11287	11953	12569	13183	13859	14551	15187	15787	16477	17167	17837	18443
10007	10639	11299	11959	12577	13187	13873	14557	15193	15791	16481	17183	17839	18451
10009	10651	11311	11969	12583	13217	13877	14561	15199	15797	16487	17189	17851	18457
10037	10657	11317	11971	12589	13219	13879	14563	15217	15803	16493	17191	17863	18461
10039	10663	11321	11981	12601	13229	13883	14591	15227	15809	16519	17203	17881	18481
10061	10667	11329	11987	12611	13241	13901	14593	15233	15817	16529	17207	17891	18493
10067	10687	11351	12007	12613	13249	13903	14621	15241	15823	16547	17209	17903	18503
10069	10691	11353	12011	12619	13259	13907	14627	15259	15859	16553	17231	17909	18517
10079	10709	11369	12037	12637	13267	13913	14629	15263	15877	16561	17239	17911	18521
10091	10711	11383	12041	12641	13291	13921	14633	15269	15881	16567	17257	17921	18523
10093	10723	11393	12043	12647	13297	13931	14639	15271	15887	16573	17291	17923	18539
10099	10729	11399	12049	12653	13309	13933	14653	15277	15889	16603	17293	17929	18541
10103	10733	11411	12071	12659	13313	13963	14657	15287	15901	16607	17299	17939	18553
10111	10739	11423	12073	12671	13327	13967	14669	15289	15907	16619	17317	17957	18583
10133	10753	11437	12097	12689	13331	13997	14683	15299	15913	16631	17321	17959	18587
10139	10771	11443	12101	12697	13337	13999	14699	15307	15919	16633	17327	17971	18593
10141	10781	11447	12107	12703	13339	14009	14713	15313	15923	16649	17333	17977	18617
10151	10789	11467	12109	12713	13367	14011	14717	15319	15937	16651	17341	17981	18637
10159	10799	11471	12113	12721	13381	14029	14723	15329	15959	16657	17351	17987	18661
10163	10831	11483	12119	12739	13397	14033	14731	15331	15971	16661	17359	17989	18671
10169	10837	11489	12143	12743	13399	14051	14737	15349	15973	16673	17377	18013	18679
10177	10847	11491	12149	12757	13411	14057	14741	15359	15991	16691	17383	18041	18691

Continued and Conjugate Fractions

Continued Fractions.—In dealing with a cumbersome fraction, or one which does not have satisfactory factors, it may be possible to substitute some other, approximately equal, fraction which is simpler or which can be factored satisfactorily. Continued fractions provide a means of computing a series of fractions each of which is a closer approximation to the original fraction than the one preceding it in the series.

A continued fraction is a proper fraction (one whose numerator is smaller than its denominator) expressed in the form

$$\frac{N}{D} = \cfrac{1}{D_1 + \cfrac{1}{D_2 + \cfrac{1}{D_3 + \dots}}}$$

It is convenient to write the above expression as

$$\frac{N}{D} = \frac{1}{D_1} + \frac{1}{D_2} + \frac{1}{D_3} + \frac{1}{D_4} + \dots$$

The continued fraction is produced from a proper fraction N/D by dividing the numerator N both into itself and into the denominator D. Dividing the numerator into itself gives a result of 1; dividing the numerator into the denominator gives a whole number D_1 plus a remainder fraction R_1. The process is then repeated on the remainder fraction R_1 to obtain D_2 and R_2; then D_3, R_3, etc., until a remainder of zero results. As an example, using $N/D = 2153/9277$,

$$\frac{2153}{9277} = \frac{2153 \div 2153}{9277 \div 2153} = \cfrac{1}{4 + \cfrac{665}{2153}} = \frac{1}{D_1 + R_1}$$

$$R_1 = \frac{665}{2153} = \cfrac{1}{3 + \cfrac{158}{665}} = \frac{1}{D_2 + R_2} \text{ etc.}$$

from which it may be seen that $D_1 = 4$, $R_1 = 665/2153$; $D_2 = 3$, $R_2 = 158/665$; and, continuing as was explained previously, it would be found that: $D_3 = 4$, $R_3 = 33/158$; …; $D_9 = 2$, $R_9 = 0$. The complete set of continued fraction elements representing 2153/9277 may then be written as

$$\frac{2153}{9277} = \frac{1}{4} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{1} + \frac{1}{3} + \frac{1}{1} + \frac{1}{2} + \frac{1}{2}$$

$$D_1 \dots\dots\dots\dots D_5 \dots\dots\dots\dots D_9$$

By following a simple procedure, together with a table organized similar to the one below for the fraction 2153/9277, the denominators D_1, D_2, … of the elements of a continued fraction may be used to calculate a series of fractions, each of which is a successively closer approximation, called a *convergent*, to the original fraction N/D.

1) The first row of the table contains column numbers numbered from 1 through 2 plus the number of elements, $2 + 9 = 11$ in this example.

2) The second row contains the denominators of the continued fraction elements in sequence but beginning in column 3 instead of column 1 because columns 1 and 2 must be blank in this procedure.

3) The third row contains the convergents to the original fraction as they are calculated and entered. Note that the fractions 1/0 and 0/1 have been inserted into columns 1 and 2. These are two arbitrary convergents, the first equal to infinity, the second to zero, which are used to facilitate the calculations.

4) The convergent in column 3 is now calculated. To find the numerator, multiply the denominator in column 3 by the numerator of the convergent in column 2 and add the numerator of the convergent in column 1. Thus, $4 \times 0 + 1 = 1$.

5) The denominator of the convergent in column 3 is found by multiplying the denominator in column 3 by the denominator of the convergent in column 2 and adding the denominator of the convergent in column 1. Thus, $4 \times 1 + 0 = 4$, and the convergent in column 3 is then ¼ as shown in the table.

6) Finding the remaining successive convergents can be reduced to using the simple equation

$$\text{CONVERGENT}_n = \frac{(D_n)(\text{NUM}_{n-1}) + \text{NUM}_{n-2}}{(D_n)(\text{DEN}_{n-1}) + \text{DEN}_{n-2}}$$

in which n = column number in the table; D_n = denominator in column n; NUM_{n-1} and NUM_{n-2} are numerators and DEN_{n-1} and DEN_{n-2} are denominators of the convergents in the columns indicated by their subscripts; and CONVERGENT_n is the convergent in column n.

Convergents of the Continued Fraction for 2153⁄9277

Column Number, n	1	2	3	4	5	6	7	8	9	10	11
Denominator, D_n	—	—	4	3	4	4	1	3	1	2	2
Convergent_n	$\frac{1}{0}$	$\frac{0}{1}$	$\frac{1}{4}$	$\frac{3}{13}$	$\frac{13}{56}$	$\frac{55}{237}$	$\frac{68}{293}$	$\frac{259}{1116}$	$\frac{327}{1409}$	$\frac{913}{3934}$	$\frac{2153}{9277}$

Notes: The decimal values of the successive convergents in the table are alternately larger and smaller than the value of the original fraction 2153⁄9277. If the last convergent in the table has the same value as the original fraction 2153⁄9277, then *all* of the other calculated convergents are correct.

Conjugate Fractions.—In addition to finding approximate ratios by the use of continued fractions and logarithms of ratios, conjugate fractions may be used for the same purpose, independently, or in combination with the other methods.

Two fractions a/b and c/d are said to be conjugate if $ad - bc = \pm 1$. Examples of such pairs are: 0⁄1 and 1⁄1; 1⁄2 and 1⁄1; and 9⁄10 and 8⁄9. Also, *every successive pair of the convergents of a continued fraction are conjugate*. Conjugate fractions have certain properties that are useful for solving ratio problems:

1) No fraction between two conjugate fractions a/b and c/d can have a denominator smaller than either b or d.

2) A new fraction, e/f, conjugate to both fractions of a given pair of conjugate fractions, a/b and c/d, and lying between them, may be created by adding respective numerators, $a + c$, and denominators, $b + d$, so that $e/f = (a + c)/(b + d)$.

3) The denominator $f = b + d$ of the new fraction e/f is the smallest of any possible fraction lying between a/b and c/d. Thus, 17⁄19 is conjugate to both 8⁄9 and 9⁄10 and no fraction with denominator smaller than 19 lies between them. This property is important if it is desired to minimize the size of the factors of the ratio to be found.

The following example shows the steps to approximate a ratio for a set of gears to any desired degree of accuracy within the limits established for the allowable size of the factors in the ratio.

Example: Find a set of four change gears, ab/cd, to approximate the ratio 2.105399 accurate to within ± 0.0001; no gear is to have more than 120 teeth.

Step 1. Convert the given ratio R to a number r between 0 and 1 by taking its reciprocal: $1/R = 1/2.105399 = 0.4749693 = r$.

Step 2. Select a pair of conjugate fractions a/b and c/d that bracket r. The pair a/b = 0⁄1 and c/d = 1⁄1, for example, will bracket 0.4749693.

Step 3. Add the respective numerators and denominators of the conjugates 0⁄1 and 1⁄1 to create a new conjugate e/f between 0 and 1: $e/f = (a + c)/(b + d) = (0 + 1)/(1 + 1) = 1/2$.

Step 4. Since 0.4749693 lies between 0⁄1 and 1⁄2, e/f must also be between 0⁄1 and 1⁄2: $e/f = (0 + 1)/(1 + 2) = 1/3$.

Step 5. Since 0.4749693 now lies between 1⁄3 and 1⁄2, e/f must also be between 1⁄3 and 1⁄2: $e/f = (1 + 1)/(3 + 2) = 2/5$.

Step 6. Continuing as above to obtain successively closer approximations of e/f to 0.4749693, and using a handheld calculator and a scratch pad to facilitate the process, the fractions below, each of which has factors less than 120, were determined:

Fraction	Numerator Factors	Denominator Factors	Error
19⁄40	19	2 × 2 × 2 × 5	+ .000031
28⁄59	2 × 2 × 7	59	– .00039
47⁄99	47	3 × 3 × 11	– .00022
104⁄219	3 × 41	7 × 37	– .000066
142⁄299	2 × 71	13 × 23	– .000053
161⁄339	7 × 23	3 × 113	– .000043
218⁄459	2 × 109	3 × 3 × 3 × 17	– .000024
256⁄539	2 × 2 × 2 × 2 × 2 × 2 × 2 × 2	7 × 7 × 11	– .000016
370⁄779	2 × 5 × 37	19 × 41	– .0000014
759⁄1598	3 × 11 × 23	2 × 17 × 47	– .00000059

Factors for the numerators and denominators of the fractions shown above were found with the aid of the Prime Numbers and Factors tables beginning on page 4. Since in Step 1 the desired ratio of 2.105399 was converted to its reciprocal 0.4749693, all of the above fractions should be inverted. Note also that the last fraction, 759⁄1598, when inverted to become 1598⁄759, is in error from the desired value by approximately one-half the amount obtained by trial and error using earlier methods.

Using Continued Fraction Convergents as Conjugates.—Since successive convergents of a continued fraction are also conjugate, they may be used to find a series of additional fractions in between themselves. As an example, the successive convergents 55⁄237 and 68⁄293 from the table of convergents for 2153⁄9277 on page 14 will be used to demonstrate the process for finding the first few in-between ratios.

Desired Fraction *N*/*D* = 2153⁄9277 = 0.2320793

	a/*b*	*e*/*f*	*c*/*d*
(1)	55⁄237 = .2320675	[a]123⁄530 = .2320755 error = –.0000039	68⁄293 = .2320819
(2)	123⁄530 = .2320755	191⁄823 = .2320778 error = –.0000016	68⁄293 = .2320819
(3)	191⁄823 = .2320778	[a]259⁄1116 = .2320789 error = –.0000005	68⁄293 = .2320819
(4)	259⁄1116 = .2320789	327⁄1409 = .2320795 error = + .0000002	68⁄293 = .2320819
(5)	259⁄1116 = .2320789	586⁄2525 = .2320792 error = – .0000001	327⁄1409 = .2320795
(6)	586⁄2525 = .2320792	913⁄3934 = .2320793 error = – .0000000	327⁄1409 = .2320795

[a] Only these ratios had suitable factors below 120.

Step 1. Check the convergents for conjugateness: $55 \times 293 - 237 \times 68 = 16115 - 16116 = -1$ proving the pair to be conjugate.

Step 2. Set up a table as shown on the next page. The leftmost column of line (1) contains the convergent of lowest value, a/b; the rightmost the higher value, c/d; and the center column the derived value e/f found by adding the respective numerators and denominators of a/b and c/d. The error or difference between e/f and the desired value N/D, error $= N/D - e/f$, is also shown.

Step 3. On line (2), the process used on line (1) is repeated with the e/f value from line (1) becoming the new value of a/b while the c/d value remains unchanged. Had the error in e/f

been + instead of –, then *e*/*f* would have been the new *c*/*d* value and *a*/*b* would be unchanged.

Step 4. The process is continued until, as seen on line (4), the error changes sign to + from the previous –. When this occurs, the *e*/*f* value becomes the *c*/*d* value on the next line instead of *a*/*b* as previously and the *a*/*b* value remains unchanged.

Positive and Negative Numbers

The degrees on a thermometer scale extending upward from the zero point may be called *positive* and may be preceded by a plus sign; thus +5 degrees means 5 degrees above zero. The degrees below zero may be called *negative* and may be preceded by a minus sign; thus – 5 degrees means 5 degrees below zero. In the same way, the ordinary numbers 1, 2, 3, etc., which are larger than 0, are called positive numbers; but numbers can be conceived of as extending in the other direction from 0, numbers that, in fact, are less than 0, and these are called negative. As these numbers must be expressed by the same figures as the positive numbers they are designated by a minus sign placed before them, thus: (–3). A negative number should always be enclosed within parentheses whenever it is written in line with other numbers; for example: $17 + (-13) - 3 \times (-0.76)$.

Negative numbers are most commonly met with in the use of logarithms and natural trigonometric functions. The following rules govern calculations with negative numbers.

A negative number can be added to a positive number by subtracting its numerical value from the positive number.

Example: $4 + (-3) = 4 - 3 = 1$.

A negative number can be subtracted from a positive number by adding its numerical value to the positive number.

Example: $4 - (-3) = 4 + 3 = 7$.

A negative number can be added to a negative number by adding the numerical values and making the sum negative.

Example: $(-4) + (-3) = -7$.

A negative number can be subtracted from a larger negative number by subtracting the numerical values and making the difference negative.

Example: $(-4) - (-3) = -1$.

A negative number can be subtracted from a smaller negative number by subtracting the numerical values and making the difference positive.

Example: $(-3) - (-4) = 1$.

If in a subtraction the number to be subtracted is larger than the number from which it is to be subtracted, the calculation can be carried out by subtracting the smaller number from the larger, and indicating that the remainder is negative.

Example: $3-5 = -(5-3) = -2$.

When a positive number is to be multiplied or divided by a negative numbers, multiply or divide the numerical values as usual; the product or quotient, respectively,is negative. The same rule is true if a negative number is multiplied or divided by a positive number.

Examples: $4 \times (-3) = -12 \quad (-4) \times 3 = -12$

$15 \div (-3) = -5 \quad (-15) \div 3 = -5$

When two negative numbers are to be multiplied by each other, the product is positive. When a negative number is divided by a negative number, the quotient is positive.

Examples: $(-4) \times (-3) = 12$; $(-4) \div (-3) = 1.333$.

The two last rules are often expressed for memorizing as follows: “Equal signs make plus, unequal signs make minus.”

Powers, Roots, and Reciprocals

The *square* of a number (or quantity) is the product of that number multiplied by itself. Thus, the square of 9 is $9 \times 9 = 81$. The square of a number is indicated by the *exponent* (^{2}), thus: $9^2 = 9 \times 9 = 81$.

The *cube* or *third power* of a number is the product obtained by using that number as a factor three times. Thus, the cube of 4 is $4 \times 4 \times 4 = 64$, and is written 4^3.

If a number is used as a factor four or five times, respectively, the product is the fourth or fifth power. Thus, $3^4 = 3 \times 3 \times 3 \times 3 = 81$, and $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$. A number can be raised to any power by using it as a factor the required number of times.

The *square root* of a given number is that number which, when multiplied by itself, will give a product equal to the given number. The square root of 16 (written $\sqrt{16}$) equals 4, because $4 \times 4 = 16$.

The *cube root* of a given number is that number which, when used as a factor three times, will give a product equal to the given number. Thus, the cube root of 64 (written $\sqrt[3]{64}$) equals 4, because $4 \times 4 \times 4 = 64$.

The fourth, fifth, etc., roots of a given number are those numbers which when used as factors four, five, etc., times, will give as a product the given number. Thus, $\sqrt[4]{16} = 2$, because $2 \times 2 \times 2 \times 2 = 16$.

In some formulas, there may be such expressions as $(a^2)^3$ and $a^{3/2}$. The first of these, $(a^2)^3$, means that the number a is first to be squared, a^2, and the result then cubed to give a^6. Thus, $(a^2)^3$ is equivalent to a^6 which is obtained by *multiplying* the exponents 2 and 3. Similarly, $a^{3/2}$ may be interpreted as the cube of the square root of a, $(\sqrt{a})^3$, or $(a^{1/2})^3$, so that, for example, $16^{3/2} = (\sqrt{16})^3 = 64$.

The multiplications required for raising numbers to powers and the extracting of roots are greatly facilitated by the use of logarithms. Extracting the square root and cube root by the regular arithmetical methods is a slow and cumbersome operation, and any roots can be more rapidly found by using logarithms.

When the power to which a number is to be raised is not an integer, say 1.62, the use of either logarithms or a scientific calculator becomes the only practical means of solution.

The *reciprocal* R of a number N is obtained by dividing 1 by the number; $R = 1/N$. Reciprocals are useful in some calculations because they avoid the use of negative characteristics as in calculations with logarithms and in trigonometry. In trigonometry, the values *cosecant, secant,* and *cotangent* are often used for convenience and are the reciprocals of the *sine, cosine,* and *tangent*, respectively (see page 83). The reciprocal of a fraction, for instance $3/4$, is the fraction inverted, since $1 \div 3/4 = 1 \times 4/3 = 4/3$.

Powers of Ten Notation

Powers of ten notation is used to simplify calculations and ensure accuracy, particularly with respect to the position of decimal points, and also simplifies the expression of numbers which are so large or so small as to be unwieldy. For example, the metric (SI) pressure unit pascal is equivalent to 0.00000986923 atmosphere or 0.0001450377 pound/inch^2. In powers of ten notation, these figures are 9.86923×10^{-6} atmosphere and 1.450377×10^{-4} pound/inch^2. The notation also facilitates adaptation of numbers for electronic data processing and computer readout.

Expressing Numbers in Powers of Ten Notation.—In this system of notation, every number is expressed by two factors, one of which is some integer from 1 to 9 followed by a decimal and the other is some power of 10.

Thus, 10,000 is expressed as 1.0000×10^4 and 10,463 as 1.0463×10^4. The number 43 is expressed as 4.3×10 and 568 is expressed. as 5.68×10^2.

In the case of decimals, the number 0.0001, which as a fraction is $\frac{1}{10,000}$, is expressed as 1×10^{-4} and 0.0001463 is expressed as 1.463×10^{-4}. The decimal 0.498 is expressed as 4.98×10^{-1} and 0.03146 is expressed as 3.146×10^{-2}.

Rules for Converting Any Number to Powers of Ten Notation.—Any number can be converted to the powers of ten notation by means of one of two rules.

Rule 1: If the number is a whole number or a whole number and a decimal so that it has digits to the left of the decimal point, the decimal point is moved a sufficient number of places to the *left* to bring it to the immediate right of the first digit. With the decimal point shifted to this position, the number so written comprises the *first* factor when written in powers of ten notation.

The number of places that the decimal point is moved to the left to bring it immediately to the right of the first digit is the *positive* index or power of 10 that comprises the *second* factor when written in powers of ten notation.

Thus, to write 4639 in this notation, the decimal point is moved three places to the left giving the two factors: 4.639×10^3. Similarly,

$$431.412 = 4.31412 \times 10^2$$

$$986388 = 9.86388 \times 10^5$$

Rule 2: If the number is a decimal, i.e., it has digits entirely to the right of the decimal point, then the decimal point is moved a sufficient number of places to the *right* to bring it immediately to the right of the first digit. With the decimal point shifted to this position, the number so written comprises the *first* factor when written in powers of ten notation.

The number of places that the decimal point is moved to the *right* to bring it immediately to the right of the first digit is the *negative* index or power of 10 that follows the number when written in powers of ten notation.

Thus, to bring the decimal point in 0.005721 to the immediate right of the first digit, which is 5, it must be moved *three* places to the right, giving the two factors: 5.721×10^{-3}. Similarly,

$$0.469 = 4.69 \times 10^{-1}$$

$$0.0000516 = 5.16 \times 10^{-5}$$

Multiplying Numbers Written in Powers of Ten Notation.—When multiplying two numbers written in the powers of ten notation together, the procedure is as follows:

1) Multiply the first factor of one number by the first factor of the other to obtain the first factor of the product.

2) Add the index of the second factor (which is some power of 10) of one number to the index of the second factor of the other number to obtain the index of the second factor (which is some power of 10) in the product. Thus:

$$(4.31 \times 10^{-2}) \times (9.0125 \times 10) = (4.31 \times 9.0125) \times 10^{-2+1} = 38.844 \times 10^{-1}$$

$$(5.986 \times 10^4) \times (4.375 \times 10^3) = (5.986 \times 4.375) \times 10^{4+3} = 26.189 \times 10^7$$

In the preceding calculations, neither of the results shown are in the conventional powers of ten form since the first factor in each has two digits. In the conventional powers of ten notation, the results would be

$$38.844 \times 10^{-1} = 3.884 \times 10^{0} = 3.884$$

since $10^0 = 1$, and

$$26.189 \times 10^{7} = 2.619 \times 10^{8}$$

in each case rounding off the first factor to three decimal places.

When multiplying several numbers written in this notation together, the procedure is the same. All of the first factors are multiplied together to get the first factor of the product and all of the indices of the respective powers of ten are added together, taking into account their respective signs, to get the index of the second factor of the product. Thus, $(4.02 \times 10^{-3}) \times (3.987 \times 10) \times (4.863 \times 10^{5}) = (4.02 \times 3.987 \times 4.863) \times (10^{-3+1+5}) = 77.94 \times 10^{3} = 7.79 \times 10^{4}$ rounding off the first factor to two decimal places.

Dividing Numbers Written in Powers of Ten Notation.—When dividing one number by another when both are written in this notation, the procedure is as follows:

1) Divide the first factor of the dividend by the first factor of the divisor to get the first factor of the quotient.

2) Subtract the index of the second factor of the divisor from the index of the second factor of the dividend, taking into account their respective signs, to get the index of the second factor of the quotient. Thus:

$$(4.31 \times 10^{-2}) \div (9.0125 \times 10) =$$

$$(4.31 \div 9.0125) \times (10^{-2-1}) = 0.4782 \times 10^{-3} = 4.782 \times 10^{-4}$$

It can be seen that this system of notation is helpful where several numbers of different magnitudes are to be multiplied and divided.

Example: Find the quotient of $\dfrac{250 \times 4698 \times 0.00039}{43678 \times 0.002 \times 0.0147}$

Solution: Changing all these numbers to powers of ten notation and performing the operations indicated:

$$\frac{(2.5 \times 10^{2}) \times (4.698 \times 10^{3}) \times (3.9 \times 10^{-4})}{(4.3678 \times 10^{4}) \times (2 \times 10^{-3}) \times (1.47 \times 10^{-2})} =$$

$$= \frac{(2.5 \times 4.698 \times 3.9)(10^{2+3-4})}{(4.3678 \times 2 \times 1.47)(10^{4-3-2})} = \frac{45.8055 \times 10}{12.8413 \times 10^{-1}}$$

$$= 3.5670 \times 10^{1-(-1)}$$

$$= 3.5670 \times 10^{2}$$

$$= 356.70$$

Preferred Numbers

American National Standard for Preferred Numbers.—This ANSI Standard Z17.1-1973 covers basic series of preferred numbers which are independent of any measurement system and therefore can be used with metric or customary units.

The numbers are rounded values of the following five geometric series of numbers: $10^{N/5}$, $10^{N/10}$, $10^{N/20}$, $10^{N/40}$, and $10^{N/80}$, where N is an integer in the series 0, 1, 2, 3, etc. The designations used for the five series are respectively R5, R10, R20, R40, and R80.

The R5 series gives 5 numbers approximately 60 per cent apart, the R10 series gives 10 numbers approximately 25 per cent apart, the R20 series gives 20 numbers approximately 12 per cent apart, the R40 series gives 40 numbers approximately 6 per cent apart, and the R80 series gives 80 numbers approximately 3 per cent apart.

ALGEBRA AND EQUATIONS

Rearrangement and Transposition of Terms in Formulas

A formula is a rule for a calculation expressed by using letters and signs instead of writing out the rule in words; by this means, it is possible to condense, in a very small space, the essentials of long and cumbersome rules. The letters used in formulas simply stand in place of the figures that are to be substituted when solving a specific problem.

As an example, the formula for the horsepower transmitted by belting may be written

$$P = \frac{SVW}{33{,}000}$$

where P = horsepower transmitted; S = working stress of belt per inch of width in pounds; V = velocity of belt in feet per minute; and, W = width of belt in inches.

If the working stress S, the velocity V, and the width W are known, the horsepower can be found directly from this formula by inserting the given values. Assume $S = 33$; $V = 600$; and $W = 5$. Then

$$P = \frac{33 \times 600 \times 5}{33{,}000} = 3$$

Assume that the horsepower P, the stress S, and the velocity V are known, and that the width of belt, W, is to be found. The formula must then be rearranged so that the symbol W will be on one side of the equals sign and all the known quantities on the other. The rearranged formula is as follows:

$$\frac{P \times 33{,}000}{SV} = W$$

The quantities (S and V) that were in the numerator on the right side of the equals sign are moved to the denominator on the left side, and "33,000," which was in the denominator on the right side of the equals sign, is moved to the numerator on the other side. Symbols that are not part of a fraction, like "P" in the formula first given, are to be considered as being numerators (having the denominator 1).

Thus, any formula of the form $A = B/C$ can be rearranged as follows:

$$A \times C = B \qquad \text{and} \qquad C = \frac{B}{A}$$

Suppose a formula to be of the form

$$A = \frac{B \times C}{D}$$

Then $$D = \frac{B \times C}{A} \qquad \frac{A \times D}{C} = B \qquad \frac{A \times D}{B} = C$$

The method given is only directly applicable when all the quantities in the numerator or denominator are standing independently or are *factors of a product*. If connected by + or – signs, the entire numerator or denominator must be moved as a unit, thus,

Given: $$\frac{B + C}{A} = \frac{D + E}{F}$$ to solve for F

then $$\frac{F}{A} = \frac{D + E}{B + C}$$

and $$F = \frac{A(D + E)}{B + C}$$

A quantity preceded by a + or – sign can be transposed to the opposite side of the equals sign by changing its sign; if the sign is +, change it to – on the other side; if it is –, change it to +. This process is called *transposition* of terms.

Example: $B + C = A - D$ then

$$A = B + C + D$$
$$B = A - D - C$$
$$C = A - D - B$$

Sequence of Performing Arithmetic Operations

When several numbers or quantities in a formula are connected by signs indicating that additions, subtractions, multiplications, and divisions are to be made, the multiplications and divisions should be carried out first, in the sequence in which they appear, before the additions or subtractions are performed.

Example:

$$10 + 26 \times 7 - 2 = 10 + 182 - 2 = 190$$
$$18 \div 6 + 15 \times 3 = 3 + 45 = 48$$
$$12 + 14 \div 2 - 4 = 12 + 7 - 4 = 15$$

When it is required that certain additions and subtractions should precede multiplications and divisions, use is made of parentheses ( ) and brackets [ ]. These signs indicate that the calculation inside the parentheses or brackets should be carried out completely by itself before the remaining calculations are commenced. If one bracket is placed inside another, the one inside is first calculated.

Example:

$$(6 - 2) \times 5 + 8 = 4 \times 5 + 8 = 20 + 8 = 28$$
$$6 \times (4 + 7) \div 22 = 6 \times 11 \div 22 = 66 \div 22 = 3$$
$$2 + [10 \times 6(8 + 2) - 4] \times 2 = 2 + [10 \times 6 \times 10 - 4] \times 2$$
$$= 2 + [600 - 4] \times 2 = 2 + 596 \times 2 = 2 + 1192 = 1194$$

The parentheses are considered as a sign of multiplication; for example, $6(8 + 2) = 6 \times (8 + 2)$.

The line or bar between the numerator and denominator in a fractional expression is to be considered as a division sign. For example,

$$\frac{12 + 16 + 22}{10} = (12 + 16 + 22) \div 10 = 50 \div 10 = 5$$

In formulas, the multiplication sign (×) is often left out between symbols or letters, the values of which are to be multiplied. Thus,

$$AB = A \times B \quad \text{and} \quad \frac{ABC}{D} = (A \times B \times C) \div D$$

Ratio and Proportion

The *ratio* between two quantities is the quotient obtained by dividing the first quantity by the second. For example, the ratio between 3 and 12 is ¼, and the ratio between 12 and 3 is 4. Ratio is generally indicated by the sign (:); thus, 12 : 3 indicates the ratio of 12 to 3.

A *reciprocal*, or *inverse* ratio, is the opposite of the original ratio. Thus, the inverse ratio of 5 : 7 is 7 : 5.

In a *compound* ratio, each term is the product of the corresponding terms in two or more simple ratios. Thus, when

$$8:2 = 4 \qquad 9:3 = 3 \qquad 10:5 = 2$$

then the compound ratio is

$$8 \times 9 \times 10 : 2 \times 3 \times 5 = 4 \times 3 \times 2$$

$$720 : 30 = 24$$

Proportion is the equality of ratios. Thus,

$$6:3 = 10:5 \qquad \text{or} \qquad 6:3::10:5$$

The first and last terms in a proportion are called the *extremes;* the second and third, the *means*. The product of the extremes is equal to the product of the means. Thus,

$$25:2 = 100:8 \qquad \text{and} \qquad 25 \times 8 = 2 \times 100$$

If three terms in a proportion are known, the remaining term may be found by the following rules:

The first term is equal to the product of the second and third terms, divided by the fourth.

The second term is equal to the product of the first and fourth terms, divided by the third.

The third term is equal to the product of the first and fourth terms, divided by the second.

The fourth term is equal to the product of the second and third terms, divided by the first.

Example: Let x be the term to be found, then,

$$x : 12 = 3.5 : 21 \qquad x = \frac{12 \times 3.5}{21} = \frac{42}{21} = 2$$

$$\tfrac{1}{4} : x = 14 : 42 \qquad x = \frac{\tfrac{1}{4} \times 42}{14} = \frac{1}{4} \times 3 = \frac{3}{4}$$

$$5 : 9 = x : 63 \qquad x = \frac{5 \times 63}{9} = \frac{315}{9} = 35$$

$$\tfrac{1}{4} : \tfrac{7}{8} = 4 : x \qquad x = \frac{\tfrac{7}{8} \times 4}{\tfrac{1}{4}} = \frac{3\tfrac{1}{2}}{\tfrac{1}{4}} = 14$$

If the second and third terms are the same, that number is the *mean proportional* between the other two. Thus, $8 : 4 = 4 : 2$, and 4 is the mean proportional between 8 and 2. The mean proportional between two numbers may be found by multiplying the numbers together and extracting the square root of the product. Thus, the mean proportional between 3 and 12 is found as follows:

$$3 \times 12 = 36 \qquad \text{and} \qquad \sqrt{36} = 6$$

which is the mean proportional.

Practical Examples Involving Simple Proportion.—If it takes 18 days to assemble 4 lathes, how long would it take to assemble 14 lathes?

Let the number of days to be found be x. Then write out the proportion as follows:

$$4:18 = 14:x$$

$$(\text{lathes : days} = \text{lathes : days})$$

Now find the fourth term by the rule given:

$$x = \frac{18 \times 14}{4} = 63 \text{ days}$$

Thirty-four linear feet of bar stock are required for the blanks for 100 clamping bolts. How many feet of stock would be required for 912 bolts?

Let x = total length of stock required for 912 bolts.

$$34:100 = x:912$$

$$(\text{feet : bolts} = \text{feet : bolts})$$

Then, the third term $x = (34 \times 912)/100 = 310$ feet, approximately.

Inverse Proportion.—In an inverse proportion, as one of the items involved *increases*, the corresponding item in the proportion *decreases*, or vice versa. For example, a factory employing 270 men completes a given number of typewriters weekly, the number of working hours being 44 per week. How many men would be required for the same production if the working hours were reduced to 40 per week?

The time per week is in an inverse proportion to the number of men employed; the shorter the time, the more men. The inverse proportion is written:

$$270 : x = 40 : 44$$

(men, 44-hour basis: men, 40-hour basis = time, 40-hour basis: time, 44-hour basis) Thus

$$\frac{270}{x} = \frac{40}{44} \qquad \text{and} \qquad x = \frac{270 \times 44}{40} = 297 \text{ men}$$

Problems Involving Both Simple and Inverse Proportions.—If two groups of data are related both by direct (simple) and inverse proportions among the various quantities, then a simple mathematical relation that may be used in solving problems is as follows:

$$\frac{\text{Product of all directly proportional items in first group}}{\text{Product of all inversely proportional items in first group}}$$

$$= \frac{\text{Product of all directly proportional items in second group}}{\text{Product of all inversely proportional items in second group}}$$

Example: If a man capable of turning 65 studs in a day of 10 hours is paid $6.50 per hour, how much per hour ought a man be paid who turns 72 studs in a 9-hour day, if compensated in the same proportion?

The first group of data in this problem consists of the number of hours worked by the first man, his hourly wage, and the number of studs which he produces per day; the second group contains similar data for the second man except for his unknown hourly wage, which may be indicated by *x*.

The labor cost per stud, as may be seen, is directly proportional to the number of hours worked and the hourly wage. These quantities, therefore, are used in the numerators of the fractions in the formula. The labor cost per stud is inversely proportional to the number of studs produced per day. (The greater the number of studs produced in a given time the less the cost per stud.) The numbers of studs per day, therefore, are placed in the denominators of the fractions in the formula. Thus,

$$\frac{10 \times 6.50}{65} = \frac{9 \times x}{72}$$

$$x = \frac{10 \times 6.50 \times 72}{65 \times 9} = \$8.00 \text{ per hour}$$

Percentage

If out of 100 pieces made, 12 do not pass inspection, it is said that 12 per cent (12 of the hundred) are rejected. If a quantity of steel is bought for $100 and sold for $140, the profit is 28.6 per cent of the selling price.

The per cent of gain or loss is found by dividing the amount of gain or loss by the *original* number of which the percentage is wanted, and multiplying the quotient by 100.

Example: Out of a total output of 280 castings a day, 30 castings are, on an average, rejected. What is the percentage of bad castings?

$$\frac{30}{280} \times 100 = 10.7 \text{ per cent}$$

If by a new process 100 pieces can be made in the same time as 60 could formerly be made, what is the gain in output of the new process over the old, expressed in per cent?

Original number, 60; gain 100 – 60 = 40. Hence,

$$\frac{40}{60} \times 100 = 66.7 \text{ per cent}$$

Care should be taken always to use the original number, or the number of which the percentage is wanted, as the divisor in all percentage calculations. In the example just given, it is the percentage of gain over the old output 60 that is wanted and not the percentage with relation to the new output too. Mistakes are often made by overlooking this important point.

Interest

Interest is money paid for the use of money lent for a certain time. *Simple* interest is the interest paid on the principal (money lent) only. When simple interest that is due is not paid, and its amount is added to the interest-bearing principal, the interest calculated on this new principal is called *compound* interest. The compounding of the interest into the principal may take place yearly or more often, according to circumstances.

Interest Formulas.—The symbols used in the formulas to calculate various types of interest are:

P = principal or amount of money lent

I = nominal annual interest rate stated as a percentage, i.e., 10 per cent per annum

I_e = effective annual interest rate when interest is compounded more often than once a year (see *Nominal vs. Effective Interest Rates*)

i = nominal annual interest rate per cent expressed as a decimal, i.e., if I = 12 per cent, then i = 12⁄100 = 0.12

n = number of annual interest periods

m = number of interest compounding periods in one year

S = a sum of money at the end of n interest periods from the present date that is equivalent to P with added interest i

R = the payment at the end of each period in a uniform series of payments continuing for n periods, the entire series equivalent to P at interest rate i

Note: The exact amount of interest for one day is 1⁄365 of the interest for one year. Banks, however, customarily take the year as composed of 12 months of 30 days, making a total of 360 days to a year. This method is also used for home-mortgage-type payments, so that the interest rate per month is 30⁄360 = 1⁄12 of the annual interest rate. For example, if I is a 12 per cent per annum nominal interest rate, then for a 30-day period, the interest rate is (12 × 1⁄12) = 1.0 per cent per month. The decimal rate per month is then 1.0⁄100 = 0.01.

Simple Interest.—The formulas for simple interest are:

$$\text{Interest for } n \text{ years} = Pin$$

$$\text{Total amount after } n \text{ years, } S = P + Pin$$

Example: For $250 that has been lent for three years at 6 per cent simple interest: P = 250; I = 6; i = I/100 = 0.06; n = 3.

$$S = 250 + (250 \times 0.06 \times 3) = 250 + 45 = \$295$$

Compound Interest.—The following formulas apply when compound interest is to be computed and assuming that the interest is compounded annually.

$$S = P(1+i)^n$$
$$P = S/(1+i)^n$$
$$i = (S/P)^{1/n} - 1$$
$$n = (\log S - \log P)/\log(1+i)$$

Example: At 10 per cent interest compounded annually for 10 years, a principal amount P of \$1000 becomes a sum S of

$$S = 1000(1 + 10/100)^{10} = \$2{,}593.74$$

If a sum S = \$2593.74 is to be accumulated, beginning with a principal P = \$1,000 over a period n = 10 years, the interest rate i to accomplish this would have to be i = $(2593.74/1000)^{1/10} - 1 = 0.09999$, which rounds to 0.1, or 10 per cent.

For a principal P = \$500 to become S = \$1,000 at 6 per cent interest compounded annually, the number of years n would have to be

$$n = (\log 1000 - \log 500)/\log(1 + 0.06)$$
$$= (3 - 2.69897)/0.025306 = 11.9 \text{ years}$$

To triple the principal P = \$500 to become S = \$1,500, the number of years would have to be

$$n = (\log 1500 - \log 500)/\log(1 + 0.06)$$
$$= (3.17609 - 2.69897)/0.025306 = 18.85 \text{ years}$$

Interest Compounded More Often Than Annually.—If interest is payable m times a year, it will be computed m times during each year, or nm times during n years. The rate for each compounding period will be i/m if i is the nominal annual decimal interest rate. Therefore, at the end of n years, the amount S will be: $S = P(1 + i/m)^{nm}$.

As an example, if P = \$1,000; n is 5 years, the interest payable quarterly, and the annual rate is 6 per cent, then $n = 5$; $m = 4$; $i = 0.06$; $i/m = 0.06/4 = 0.015$; and $nm = 5 \times 4 = 20$, so that

$$S = 1000(1 + 0.015)^{20} = \$1{,}346.86$$

Nominal vs. Effective Interest Rates.—Deposits in savings banks, automobile loans, interest on bonds, and many other transactions of this type involve computation of interest due and payable more often than once a year. For such instances, there is a difference between the *nominal* annual interest rate stated to be the cost of borrowed money and the *effective* rate that is actually charged.

For example, a loan with interest charged at 1 per cent per month is described as having an interest rate of 12 per cent per annum. To be precise, this rate should be stated as being a *nominal* 12 per cent per annum compounded monthly; the actual or *effective* rate for monthly payments is 12.7 per cent. For quarterly compounding, the effective rate would be 12.6 per cent:

$$I_e = (1 + I/m)^m - 1$$

In this formula, I_e is the effective annual rate, I is the nominal annual rate, and m is the number of times per year the money is compounded.

Example: For a nominal per annum rate of 12 per cent, with monthly compounding, the effective per annum rate is

$$I_e = (1 + 0.12/12)^{12} - 1 = 0.1268 = 12.7 \text{ per cent effective per annum rate}$$

Example: Same as before but with quarterly compounding:

$$I_e = (1 + 0.12/4)^4 - 1 = 0.1255 = 12.6 \text{ per cent effective per annum rate}$$

Example: Same as before but with annual compounding.

$$I_e = (1 + 0.12/1)^1 - 1 = 0.12 = 12 \text{ per cent effective per annum rate}$$

This last example shows that for once-a-year-compounding, the nominal and effective per annum rates are identical.

Finding Unknown Interest Rates.—If a single payment of P dollars is to produce a sum of S dollars after n annual compounding periods, the per annum decimal interest rate is found using:

$$i = \sqrt[n]{\frac{S}{P}} - 1$$

Present Value and Discount.—The present value or present worth P of a given amount S is the amount P that, when placed at interest i for a given time n, will produce the given amount S.

$$\text{At simple interest, } P = S/(1 + ni)$$

$$\text{At compound interest, } P = S/(1 + i)^n$$

The *true discount D* is the difference between S and P: $D = S - P$.

These formulas are for an annual interest rate. If interest is payable other than annually, modify the formulas as indicated in the formulas in the section *Interest Compounded More Often Than Annually*.

Example: Required the present value and discount of $500 due in six months at 6 per cent simple interest. Here, $S = 500$; $n = 6/12 = 0.5$ year; $i = 0.06$. Then, $P = 500/(1 + 0.5 \times 0.06)$ = $485.44.

Required the sum that, placed at 5 per cent compound interest, will in three years produce $5,000. Here, $S = 5000$; $i = 0.05$; $n = 3$. Then,

$$P = 5000/(1 + 0.05)^3 = \$4,319.19$$

Annuities.—An annuity is a fixed sum paid at regular intervals. In the formulas that follow, yearly payments are assumed. It is customary to calculate annuities on the basis of compound interest. If an annuity A is to be paid out for n consecutive years, the interest rate being i, then the present value P of the annuity is

$$P = A\frac{(1 + i)^n - 1}{i(1 + i)^n}$$

Example: If an annuity of $200 is to be paid for 10 years, what is the present amount of money that needs to be deposited if the interest is 5 per cent. Here, $A = 200$; $i = 0.05$; $n = 10$:

$$P = 200\frac{(1 + 0.05)^{10} - 1}{0.05(1 + 0.05)^{10}} = \$1,544.35$$

The annuity a principal P drawing interest at the rate i will give for a period of n years is

$$A = P\frac{i(1 + i)^n}{(1 + i)^n - 1}$$

Example: A sum of $10,000 is placed at 4 per cent. What is the amount of the annuity payable for 20 years out of this sum: Here, $P = 10000$; $i = 0.04$; $n = 20$:

$$A = 10,000\frac{0.04(1 + 0.04)^{20}}{(1 + 0.04)^{20} - 1} = \$735.82$$

If at the *beginning* of each year a sum A is set aside at an interest rate i, the total value S of the sum set aside, with interest, at the end of n years, will be

$$S = A\frac{(1+i)[(1+i)^n - 1]}{i}$$

If at the *end* of each year a sum A is set aside at an interest rate i, then the total value S of the principal, with interest, at the end of n years will be

$$S = A\frac{(1+i)^n - 1}{i}$$

If a principal P is increased or decreased by a sum A at the end of each year, then the value of the principal after n years will be

$$S = P(1+i)^n \pm A\frac{(1+i)^n - 1}{i}$$

If the sum A by which the principal P is decreased each year is greater than the total yearly interest on the principal, then the principal, with the accumulated interest, will be entirely used up in n years:

$$n = \frac{\log A - \log(A - iP)}{\log(1+i)}$$

Sinking Funds.—Amortization is "the extinction of debt, usually by means of a sinking fund." The sinking fund is created by a fixed investment R placed each year at compound interest for a term of years n, and is therefore an annuity of sufficient size to produce at the end of the term of years the amount S necessary for the repayment of the principal of the debt, or to provide a definite sum for other purposes. Then,

$$S = R\frac{(1+i)^n - 1}{i} \qquad \text{and} \qquad R = S\frac{i}{(1+i)^n - 1}$$

Example: If $2,000 is invested annually for 10 years at 4 per cent compound interest, as a sinking fund, what would be the total amount of the fund at the expiration of the term? Here, $R = 2000$; $n = 10$; $i = 0.04$:

$$S = 2000\frac{(1+0.04)^{10} - 1}{0.04} = \$24{,}012.21$$

Evaluating Investments in Industrial Assets

Investment in industrial assets such as machine tools, processing equipment, and other means of production may not be attractive unless the cost of such investment can be recovered with interest. The interest, or *rate of return,* should be equal to, or greater than, some specified minimum rate for each of such investments. Three methods used in analyzing prospective investments are

1) *Annual cost* of the investment at a specified minimum acceptable rate of return used as the interest rate.

2) *Present worth,* using as an interest rate a specified minimum acceptable rate of return.

3) *Prospective rate of return* compared to a specified minimum acceptable rate.

Annual Cost Method.—In the annual cost method, comparisons are made among alternative investment plans. If the annual costs in any investment plan form a non-uniform series of disbursements from year to year, a much used method for reducing *all* comparisons to an equivalent basis is as follows: Take each of the annual disbursements and use the *present worth* method developed in the next section to bring all annual costs down to a common present worth date, usually called "year 0." When this has been done, each present worth is converted to an equivalent uniform annual series of disbursements using the applicable formulas from Table 1.

Table 1. Summary of Useful Interest Formulas

The meanings of the symbols P, R, S, L, i, n, and F are as follows:

P = Principal sum of money at the present time. Also the present worth of a future payment in a series of equal payments

R = Single payment in a series of n equal payments made at the end of each interest period

S = A sum, after n interest period, equal to the compound amount of a principal sum, P, or the sum of the compound amounts of the payments, R, at interest rate i

i = Nominal annual interest rate expressed as a decimal

n = Number of interest periods, usually annual

L = Salvage value of an asset at the end of its projected useful life

$F = (1 + i)^n$ for which values are tabulated in Table 2

To Find	Given	Formula	
Simple interest, Sum	P, find S	$S = P(1 + ni)$	(1)
Single payment, Compound-amount	P, find S	$S = PF$	(2)
Single payment, Present-worth	S, find P	$P = \frac{S}{F}$	(3)
Equal-payment series, Compound-amount	R, find S	$S = R\frac{(F-1)}{i}$	(4)
Equal-payment series, Sinking-fund	S, find R	$R = S\frac{i}{(F-1)}$	(5)
Equal-payment series, Present-worth	R, find P	$P = R\frac{(F-1)}{iF}$	(6)
Equal-payment series, Capital-recovery	P, find R	$R = P\frac{iF}{(F-1)}$	(7)
Equal-payment series with salvage value, Capital-recovery	P and L, find R	$R = (P-L)\frac{iF}{(F-1)} + Li$	(8)

Example 1 (Annual Cost Calculations): An investment of $15,000 is being considered to reduce labor and labor-associated costs in a materials handling operation from $8,200 a year to $3,300. This operation is expected to be used for 10 years before being changed or discontinued entirely. In addition to the initial investment of $15,000 and the annual cost of $3,300 for labor, there are additional annual costs for power, maintenance, insurance, and property taxes of $1,800 associated with the revised operation. Based on comparisons of annual costs, should the $15,000 investment be made or the present operation continued?

The present annual cost of the operation is $8,200 for labor and labor-associated costs. The proposed operation has an annual cost of $3,300 for labor and labor extras plus $1,800 for additional power, maintenance, insurance, and taxes, plus the annual cost of recovering the initial investment of $15,000 at some interest rate (minimum acceptable rate of return).

Assuming that 10 per cent would be an acceptable rate of return on this investment over a period of 10 years, the annual amount to be recovered on the initial investment would be $15,000 multiplied by the capital recovery factor calculated using Formula (7) in Table 1. From Table 2, the factor F for 10 per cent and 10 years is seen to be 2.594.

Putting this value into Formula (7) gives:

$$R = P\frac{iF}{F-1} = 15{,}000\frac{0.1 \times 2.594}{2.594 - 1} = \$2{,}442$$

Adding this amount to the $5,100 annual cost associated with the investment ($3,300+ $1,800 = $5,100) gives a total annual cost of $7,542, which is less than the present annual cost of $8,200. Thus, the investment is justified unless there are other considerations such as the effects of income taxes, salvage values, expected life, uncertainty about the required rate of return, changes in the cost of borrowed funds, and others.

A tabulation of annual costs of alternative plans A, B, C, etc., is a good way to compare costs item by item. For Example 1:

	Item	Plan A	Plan B
1	Labor and labor extras	$8,200.00	$3,300.00
2	Annual cost of $15,000 investment using Formula (7), Table 2		2,442.00
3	Power		400.00
4	Maintenance		1,100.00
5	Property taxes and insurance		300.00
	Total annual cost	$8,200.00	$7,542.00

Example 2 (Annual Cost Considering Salvage Value): If in Example 1 the salvage value of the equipment installed was $5,000 at the end of 10 years, what effect does this have on the annual cost of the proposed investment of $15,000?

The only item in the annual cost of Example 1 that will be affected is the capital recovery amount of $2,442. The following formula gives the amount of annual capital recovery when salvage value is considered:

$$R = (P-L)\frac{iF}{(F-1)} + Li = (15{,}000-5{,}000)\frac{0.1\times 2.594}{2.594-1} + 5{,}000\times 0.1 = \$2{,}127$$

Adding this amount to the $5,100 annual cost determined previously gives a total annual cost of $7,227, which is $315 less than the previous annual cost of $7,542 for the proposed investment.

Present Worth Method.—A present worth calculation may be described as the discounting of a future payment, or a series of payments, to a cash value on the present date based on a selected interest rate. The present date is referred to as “day 0,” or “year 0.” Initial costs are already at zero date (present worth date), so no factors need be applied to initial, or “up-front,” costs. On the other hand, if salvage values are to be considered, these must be reduced to present worth and *subtracted* from the present worth of the initial investment.

The present worths of each of the alternative investments are then compared to find the lowest cost alternative. The present worth of the lowest cost alternative may then be converted to a uniform series of annual costs and these annual costs compared with an existing, in-place, annual cost. Present worth calculations are often referred to as *discounted cash flow* because this term describes both the data required and the method of calculation. *Cash flow* refers to the requirement that data must be supplied in the form of amounts and dates of receipts and payouts, and *discounted* refers to the calculation of the present worth of each of one or more future payments. The rate of return used in such calculations should be the minimum attractive interest rate before taxes.

Example 3 (Present Worth Calculation): Present worths are calculated as of the zero date of the payments being compared. Up-front (zero-day) disbursements are already at their present worth and no interest factors should be applied to them; the present worths of salvage values are *subtracted* to get the present worth of the net disbursements because the present worth of a salvage value, in effect, reduces the amount of required initial disbursements.

A) Find the present value of a salvage value of $1000 from the sale of equipment after 10 years if the expected rate of return (interest rate) is 10 per cent? Using Formula (3) from Table 1 and the value of $F = 2.594$ for $n = 10$ years and $i = 0.1$ from Table 2, the present worth $P = 1000/2.594 = \$385.51$.

B) In Example 1, the annual cost of an investment of $15,000 at 10 per cent over 10 years was $2,442. Convert this annual outlay back to its present worth.

Using Formula (6) and the value 2.594 from Table 2, $P = 2{,}442 \times (2.594 - 1)/(0.1 \times 2.594) = \$15{,}005$, which rounds to \$15,000.

Table 2. Values of $F = (1 + i)^n$ for Selected Rates of Interest, i, and Number of Annual Interest Periods, n

Number of Years, n	Annual Interest Rate, i, Expressed as a Decimal											
	0.050	0.060	0.070	0.080	0.090	0.100	0.110	0.120	0.130	0.140	0.150	0.160
	Factor $F = (1 + i)^n$											
1	1.050	1.060	1.070	1.080	1.090	1.100	1.110	1.120	1.130	1.140	1.150	1.160
2	1.103	1.124	1.145	1.166	1.188	1.210	1.232	1.254	1.277	1.300	1.323	1.346
3	1.158	1.191	1.225	1.260	1.295	1.331	1.368	1.405	1.443	1.482	1.521	1.561
4	1.216	1.262	1.311	1.360	1.412	1.464	1.518	1.574	1.630	1.689	1.749	1.811
5	1.276	1.338	1.403	1.469	1.539	1.611	1.685	1.762	1.842	1.925	2.011	2.100
6	1.340	1.419	1.501	1.587	1.677	1.772	1.870	1.974	2.082	2.195	2.313	2.436
7	1.407	1.504	1.606	1.714	1.828	1.949	2.076	2.211	2.353	2.502	2.660	2.826
8	1.477	1.594	1.718	1.851	1.993	2.144	2.305	2.476	2.658	2.853	3.059	3.278
9	1.551	1.689	1.838	1.999	2.172	2.358	2.558	2.773	3.004	3.252	3.518	3.803
10	1.629	1.791	1.967	2.159	2.367	2.594	2.839	3.106	3.395	3.707	4.046	4.411
11	1.710	1.898	2.105	2.332	2.580	2.853	3.152	3.479	3.836	4.226	4.652	5.117
12	1.796	2.012	2.252	2.518	2.813	3.138	3.498	3.896	4.335	4.818	5.350	5.936
13	1.886	2.133	2.410	2.720	3.066	3.452	3.883	4.363	4.898	5.492	6.153	6.886
14	1.980	2.261	2.579	2.937	3.342	3.797	4.310	4.887	5.535	6.261	7.076	7.988
15	2.079	2.397	2.759	3.172	3.642	4.177	4.785	5.474	6.254	7.138	8.137	9.266
16	2.183	2.540	2.952	3.426	3.970	4.595	5.311	6.130	7.067	8.137	9.358	10.748
17	2.292	2.693	3.159	3.700	4.328	5.054	5.895	6.866	7.986	9.276	10.761	12.468
18	2.407	2.854	3.380	3.996	4.717	5.560	6.544	7.690	9.024	10.575	12.375	14.463
19	2.527	3.026	3.617	4.316	5.142	6.116	7.263	8.613	10.197	12.056	14.232	16.777
20	2.653	3.207	3.870	4.661	5.604	6.727	8.062	9.646	11.523	13.743	16.367	19.461
21	2.786	3.400	4.141	5.034	6.109	7.400	8.949	10.804	13.021	15.668	18.822	22.574
22	2.925	3.604	4.430	5.437	6.659	8.140	9.934	12.100	14.714	17.861	21.645	26.186
23	3.072	3.820	4.741	5.871	7.258	8.954	11.026	13.552	16.627	20.362	24.891	30.376
24	3.225	4.049	5.072	6.341	7.911	9.850	12.239	15.179	18.788	23.212	28.625	35.236
25	3.386	4.292	5.427	6.848	8.623	10.835	13.585	17.000	21.231	26.462	32.919	40.874
26	3.556	4.549	5.807	7.396	9.399	11.918	15.080	19.040	23.991	30.167	37.857	47.414
27	3.733	4.822	6.214	7.988	10.245	13.110	16.739	21.325	27.109	34.390	43.535	55.000
28	3.920	5.112	6.649	8.267	11.167	14.421	18.580	23.884	30.633	39.204	50.066	63.800
29	4.116	5.418	7.114	9.317	12.172	15.863	20.624	26.750	34.616	44.693	57.575	74.009
30	4.322	5.743	7.612	10.063	13.268	17.449	22.892	29.960	39.116	50.950	66.212	85.850

Prospective Rate of Return Method (Discounted Cash Flow).—This method of calculating the prospective return on an investment has variously been called the *discounted cash flow method, the Investor's Method,* the *Profitability Index,* and the *interest rate of return,* but *discounted cash flow* is the most common terminology in industry.

The term "discounted cash flow" is most descriptive of the process because "cash flow" describes the amounts and dates of the receipts and disbursements and "discounted" refers to the calculation of present worth. Calculating the present worth of future payments is often described as discounting the payments to the present "zero" date.

The process is best illustrated by an example. The data in Table 3 are a tabulation of the cash flows by amount and year for two different plans of investment, the fourth column showing the differences in cash flows for each year and the differences in the totals. This type of information could equally well represent a comparison involving a choice between two different machine tools, investments in rental properties, or any other situation where the difference between two or more alternative investments are to be evaluated.

The differences in cash flows in the fourth column of Table 3 consist of a disbursement of $15,000 at date zero and receipts of $3,000 per year for 10 years. The rate of return *on the net cash flow* can be calculated from these data using the principal that *the rate of return is that interest rate at which the present worth of the net cash flow is zero*.

In this example, the present worth of the *net* cash flow will be 0 if the present worth of the $15,000 disbursements is numerically equal to the present worth of the uniform annual series of receipts of $3,000. For the disbursement of $15,000, the present worth is already $15,000, because it was made on date 0. The uniform annual series of receipts of $3,000 a year for 10 years must be converted to present worth using Formula (6) from Table 1. The interest rate needed to calculate the factor F in Formula (6) to get the necessary present worth conversion factor is not known, so that a series of trial-and-error substitutions of assumed interest rates must be made to find the correct present value factor. As a first guess, assume an interest rate of 15 per cent. Then, from Table 2, for 15 per cent and 10 years, $F = 4.046$, and substituting in Formula (6) of Table 1,

$$P = 3000\frac{4.046 - 1}{0.15 \times 4.046} = 15{,}056 \textit{ nearly}$$

so that the present worth of the net cash flow at 15 per cent interest (–$15,000 + $15,056 = $56) is slightly more than 0. If a 16 per cent interest rate is tried, P is calculated as $14,499, which is too small by $501 and is about 10 times larger than the previous difference of $56. By interpolation, the interest rate should be approximately 15.1 per cent (0.1 per cent above 15 per cent, or 0.9 per cent less than 16 per cent). Then $F = (1 + 0.151)^{10} = 4.0809$, and $P = 3{,}000(F - 1)/(0.151 \times 4.0809) = 3{,}000(4.0809 - 1)/(0.151 \times 4.0809) = 14{,}999$, *nearly*. Thus, the present worth of the net cash flow at 15.1 per cent (–$15,000 + $14,999 = –$1) is only slightly less than 0, and the prospective rate of return may be taken as 15.1 per cent.

Table 3. Comparison of Cash Flows for Two Competing Plans

Year	Annual Costs Plan A	Annual Costs Plan B	Net Cash Flow B — A
0		-$15,000	-$15,000
1	-$8,000	-5,000	+3,000
2	-8,000	-5,000	+3,000
3	-8,000	-5,000	+3,000
4	-8,000	-5,000	+3,000
5	-8,000	-5,000	+3,000
6	-8,000	-5,000	+3,000
7	-8,000	-5,000	+3,000
8	-8,000	-5,000	+3,000
9	-8,000	-5,000	+3,000
10	-8,000	-5,000	+3,000
Totals	-$80,000	-$65,000	+$15,000

Principal Algebraic Expressions and Formulas

$$a \times a = aa = a^2$$
$$a \times a \times a = aaa = a^3$$
$$a \times b = ab$$
$$a^2 b^2 = (ab)^2$$
$$a^2 a^3 = a^{2+3} = a^5$$
$$a^4 \div a^3 = a^{4-3} = a$$
$$a^0 = 1$$
$$a^2 - b^2 = (a+b)(a-b)$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

$$\frac{a^3}{b^3} = \left(\frac{a}{b}\right)^3$$
$$\frac{1}{a^3} = \left(\frac{1}{a}\right)^3 = a^{-3}$$
$$(a^2)^3 = a^{2 \times 3} = (a^3)^2 = a^6$$
$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$
$$(a+b)^3 = a^3 + 3a^2 b + 3ab^2 + b^3$$
$$(a-b)^3 = a^3 - 3a^2 b + 3ab^2 - b^3$$

$$\sqrt{a} \times \sqrt{a} = a$$
$$\sqrt[3]{a} \times \sqrt[3]{a} \times \sqrt[3]{a} = a$$
$$(\sqrt[3]{a})^3 = a$$
$$\sqrt[3]{a^2} = (\sqrt[3]{a})^2 = a^{2/3}$$
$$\sqrt[4]{\sqrt[3]{a}} = \sqrt[4 \times 3]{a} = \sqrt[3]{\sqrt[4]{a}}$$
$$\sqrt{a} + \sqrt{b} = \sqrt{a + b + 2\sqrt{ab}}$$

$$\sqrt[3]{ab} = \sqrt[3]{a} \times \sqrt[3]{b}$$
$$\sqrt[3]{\frac{a}{b}} = \frac{\sqrt[3]{a}}{\sqrt[3]{b}}$$
$$\sqrt[3]{\frac{1}{a}} = \frac{1}{\sqrt[3]{a}} = a^{-1/3}$$

When	$a \times b = x$	then	$\log a + \log b = \log x$
	$a \div b = x$	then	$\log a - \log b = \log x$
	$a^3 = x$	then	$3 \log a = \log x$
	$\sqrt[3]{a} = x$	then	$\frac{\log a}{3} = \log x$

Equations

An equation is a statement of equality between two expressions, as $5x = 105$. The unknown quantity in an equation is generally designated by the letter x. If there is more than one unknown quantity, the others are designated by letters also selected at the end of the alphabet, as y, z, u, t, etc.

An equation of the first degree is one which contains the unknown quantity only in the first power, as $3x = 9$. A quadratic equation is one which contains the unknown quantity in the second, but no higher, power, as $x^2 + 3x = 10$.

Solving Equations of the First Degree with One Unknown.—Transpose all the terms containing the unknown x to one side of the equals sign, and all the other terms to the other side. Combine and simplify the expressions as far as possible, and divide both sides by the coefficient of the unknown x. (See the rules given for transposition of formulas.)

Example:

$$22x - 11 = 15x + 10$$
$$22x - 15x = 10 + 11$$
$$7x = 21$$
$$x = 3$$

Solution of Equations of the First Degree with Two Unknowns.—The form of the simplified equations is

$$ax + by = c$$
$$a_1x + b_1y = c_1$$

Then,

$$x = \frac{cb_1 - c_1b}{ab_1 - a_1b} \qquad y = \frac{ac_1 - a_1c}{ab_1 - a_1b}$$

Example:

$$3x + 4y = 17$$
$$5x - 2y = 11$$

$$x = \frac{17 \times (-2) - 11 \times 4}{3 \times (-2) - 5 \times 4} = \frac{-34 - 44}{-6 - 20} = \frac{-78}{-26} = 3$$

The value of y can now be most easily found by inserting the value of x in one of the equations:

$$5 \times 3 - 2y = 11 \qquad 2y = 15 - 11 = 4 \qquad y = 2$$

Solution of Quadratic Equations with One Unknown.—If the form of the equation is $ax^2 + bx + c = 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example: Given the equation, $1x^2 + 6x + 5 = 0$, then $a = 1$, $b = 6$, and $c = 5$.

$$x = \frac{-6 \pm \sqrt{6^2 - 4 \times 1 \times 5}}{2 \times 1} = \frac{(-6) + 4}{2} = -1 \quad \text{or} \quad \frac{(-6) - 4}{2} = -5$$

If the form of the equation is $ax^2 + bx = c$, then

$$x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}$$

Example: A right-angle triangle has a hypotenuse 5 inches long and one side which is one inch longer than the other; find the lengths of the two sides.

Let x = one side and $x + 1$ = other side; then $x^2 + (x + 1)^2 = 5^2$ or $x^2 + x^2 + 2x + 1 = 25$; or $2x^2 + 2x = 24$; or $x^2 + x = 12$. Now referring to the basic formula, $ax^2 + bx = c$, we find that $a = 1$, $b = 1$, and $c = 12$; hence,

$$x = \frac{-1 \pm \sqrt{1 + 4 \times 1 \times 12}}{2 \times 1} = \frac{(-1) + 7}{2} = 3 \quad \text{or} \quad x = \frac{(-1) - 7}{2} = -4$$

Since the positive value (3) would apply in this case, the lengths of the two sides are $x = 3$ inches and $x + 1 = 4$ inches.

Cubic Equations.—If the given equation has the form: $x^3 + ax + b = 0$ then

$$x = \left(-\frac{b}{2} + \sqrt{\frac{a^3}{27} + \frac{b^2}{4}}\right)^{1/3} + \left(-\frac{b}{2} - \sqrt{\frac{a^3}{27} + \frac{b^2}{4}}\right)^{1/3}$$

The equation $x^3 + px^2 + qx + r = 0$, may be reduced to the form $x_1^3 + ax_1 + b = 0$ by substituting $x_1 - \frac{p}{3}$ for x in the given equation.

Series.—Some hand calculations, as well as computer programs of certain types of mathematical problems, may be facilitated by the use of an appropriate series. For example, in some gear problems, the angle corresponding to a given or calculated involute function is found by using a series together with an iterative procedure such as the Newton-Raphson method described on page 35. The following are those series most commonly used for such purposes. In the series for trigonometric functions, the angles x are in radians (1 radian = 180/π degrees). The expression $\exp(-x^2)$ means that the base e of the natural logarithm system is raised to the $-x^2$ power; e = 2.7182818.

(1)	$\sin x = x - x^3/3! + x^5/5! - x^7/7! + \cdots$	for all values of x.
(2)	$\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + \cdots$	for all values of x.
(3)	$\tan x = x + x^3/3 + 2x^5/15 + 17x^7/315 + 62x^9/2835 + \cdots$	for $\|x\| < \pi/2$.
(4)	$\arcsin x = x + x^3/6 + 1\cdot 3\cdot x^5/(2\cdot 4\cdot 5) + 1\cdot 3\cdot 5\cdot x^7/(2\cdot 4\cdot 6\cdot 7) + \cdots$	for $\|x\| \le 1$.
(5)	$\arccos x = \pi/2 - \arcsin x$	
(6)	$\arctan x = x - x^3/3 + x^5/5 - x^7/7 + \cdots$	for $\|x\| \le 1$.
(7)	$e^x = 1 + x + x^2/2! + x^3/3! + \cdots$	for all values of x.
(8)	$\exp(-x^2) = 1 - x^2 + x^4/2! - x^6/3! + \cdots$	for all values of x.
(9)	$a^x = 1 + x\log_e a + (x\log_e a)^2/2! + (x\log_e a)^3/3! + \cdots$	for all values of x.
(10)	$1/(1+x) = 1 - x + x^2 - x^3 + x^4 - \cdots$	for $\|x\| < 1$.
(11)	$1/(1-x) = 1 + x + x^2 + x^3 + x^4 + \cdots$	for $\|x\| < 1$.
(12)	$1/(1+x)^2 = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots$	for $\|x\| < 1$.
(13)	$1/(1-x)^2 = 1 + 2x + 3x^2 + 4x^3 + 5x^5 + \cdots$	for $\|x\| < 1$.
(14)	$\sqrt{(1+x)} = 1 + x/2 - x^2/(2\cdot 4) + 1\cdot 3\cdot x^3/(2\cdot 4\cdot 6) - 1\cdot 3\cdot 5\cdot x^4/(2\cdot 4\cdot 6\cdot 8) - \cdots$	for $\|x\| < 1$.
(15)	$1/(\sqrt{1+x}) = 1 - x/2 + 1\cdot 3\cdot x^2/(2\cdot 4) - 1\cdot 3\cdot 5\cdot x^3/(2\cdot 4\cdot 6) + \cdots$	for $\|x\| < 1$.
(16)	$(a+x)^n = a^n + na^{n-1}x + n(n-1)a^{n-2}x^2/2! + n(n-1)(n-2)a^{n-3}x^3/3! + \cdots$	for $x^2 < a^2$.

Derivatives of Functions.—The following are formulas for obtaining the derivatives of basic mathematical functions. In these formulas, the letter a denotes a constant; the letter x denotes a variable; and the letters u and v denote functions of the variable x. The expression d/dx means the derivative with respect to x, and as such applies to whatever expression in parentheses follows it. Thus, d/dx (ax) means the derivative with respect to x of the product (ax) of the constant a and the variable x, as given by formula (3).

To simplify the form of the formulas, the symbol D is used to represent d/dx. Thus, D is equivalent to d/dx and other forms as follows:

$$D(ax) = \frac{d(ax)}{dx} = \frac{d}{dx}(ax)$$

1) $D(a) = 0$ 2) $D(x) = 1$ 3) $D(ax) = a\cdot D(x) = a\cdot 1 = a$

4) $D(u+v) = D(u) + D(v)$ *Example:* $D(x^4 + 2x^2) = 4x^3 + 4x$

5) $D(uv) = v\cdot D(u) + u\cdot D(v)$ *Example:* $D(x^2\cdot ax^3) = ax^3\cdot 2x + x^2\cdot 3ax^2 = 5ax^4$

6) $D(u/v) = \dfrac{v\cdot D(u) - u\cdot D(v)}{v^2}$ *Example:* $D(ax^2/\sin x) = (2ax\cdot\sin x - ax^2\cdot\cos x)/\sin^2 x$

7) $D(x^n) = n\cdot x^{n-1}$ *Example:* $D(5x^7) = 35x^6$

8) $D(e^x) = e^x$

9) $D(a^x) = a^x \cdot \log_e a$ *Example:* $D(11^x) - 11^x \cdot \log_e 11$
10) $D(u^v) = v \cdot u^{v-1} \cdot D(u) + u^v \cdot \log_e u \cdot D(v)$
11) $D(\log_e x) = 1/x$
12) $D(\log_a x) = 1/x \cdot \log_e a = \log_a e/x$
13) $D(\sin x) = \cos x$ *Example:* $D(a \cdot \sin x) = a \cdot \cos x$
14) $D(\cos x) = \sin x$ 15) $D(\tan x) = \sec^2 x$

Solving Numerical Equations Having One Unknown.—The Newton-Raphson method is a procedure for solving various kinds of numerical algebraic and transcendental equations in one unknown. The steps in the procedure are simple and can be used with either a handheld calculator or as a subroutine in a computer program.

Examples of types of equations that can be solved to any desired degree of accuracy by this method are

$$f(x) = x^2 - 101 = 0, \quad f(x) = x^3 - 2x^2 - 5 = 0$$
$$\text{and} \quad f(x) = 2.9x - \cos x - 1 = 0$$

The procedure begins with an estimate, r_1, of the root satisfying the given equation. This estimate is obtained by judgment, inspection, or plotting a rough graph of the equation and observing the value r_1 where the curve crosses the x axis. This value is then used to calculate values $r_2, r_3, \ldots, r_n$ progressively closer to the exact value.

Before continuing, it is necessary to calculate the first derivative. $f'(x)$, of the function. In the above examples, $f'(x)$ is, respectively, $2x$, $3x^2 - 4x$, and $2.9 + \sin x$. These values were found by the methods described in *Derivatives of Functions* on page 34.

In the steps that follow,
r_1 is the first estimate of the value of the root of $f(x) = 0$;
$f(r_1)$ is the value of $f(x)$ for $x = r_1$;
$f'(x)$ is the first derivative of $f(x)$;
$f'(r_1)$ is the value of $f'(x)$ for $x = r_1$.

The second approximation of the root of $f(x) = 0$, r_2, is calculated from

$$r_2 = r_1 - [f(r_1)/f'(r_1)]$$

and, to continue further approximations,

$$r_n = r_{n-1} - [f(r_{n-1})/f'(r_{n-1})]$$

Example: Find the square root of 101 using Newton-Raphson methods. This problem can be restated as an equation to be solved, i.e., $f(x) = x^2 - 101 = 0$

Step 1. By inspection, it is evident that $r_1 = 10$ may be taken as the first approximation of the root of this equation. Then, $f(r_1) = f(10) = 10^2 - 101 = -1$

Step 2. The first derivative, $f'(x)$, of $x^2 - 101$ is $2x$ as stated previously, so that

$$f'(10) = 2(10) = 20.$$

Then,

$$r_2 = r_1 - f(r_1)/f'(r_1) = 10 - (-1)/20 = 10 + 0.05 = 10.05.$$

Check: $10.05^2 = 101.0025$; error = 0.0025

Step 3. The next, better approximation is

$$r_3 = r_2 - [f(r_2)/f'(r_2)] = 10.05 - [f(10.05)/f'(10.05)]$$
$$= 10.05 - [(10.05^2 - 101)/2(10.05)] = 10.049875$$

$$\mathit{Check}: 10.049875^2 = 100.9999875; \text{ error } = 0.0000125$$

Coordinate Systems

Rectangular, Cartesian Coordinates.—In a Cartesian coordinate system the coordinate axes are perpendicular to one another, and the same unit of length is chosen on the two axes. This rectangular coordinate system is used in the majority of cases.

The general form of an equation of a line in a Cartesian coordinate system is $y = mx + b$, where (x, y) is a point on the line, m is the slope (the rate at which the line is increasing or decreasing), and b is the y coordinate, the y-intercept, of the point $(0, b)$ on the y-axis where the line intersects the axis at $x = 0$.

Another form of the equation of a line is the point-slope form $(y - y_1) = m(x - x_1)$. The slope, m, is defined as a ratio of the change in the y coordinates, $y_2 - y_1$, to the change in the x coordinates, $x_2 - x_1$,

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 1: Find the general equation of a line passing through the points (3, 2) and (5, 6), and it's intersection point with the y-axis.

First, find the slope using the equation above

$$m = \frac{\Delta y}{\Delta x} = \frac{6 - 2}{5 - 3} = \frac{4}{2} = 2$$

The line has a general form of $y = 2x + b$, and the value of the constant b can be determined by substituting the coordinates of a point on the line into the general form. Using point (3,2), $2 = 2 \times 3 + b$ and rearranging, $b = 2 - 6 = -4$. As a check, using another point on the line, (5,6), yields equivalent results, $y = 6 = 2 \times 5 + b$ and $b = 6 - 10 = -4$.

The equation of the line, therefore, is $y = 2x - 4$, indicating that line $y = 2x - 4$ intersects the y-axis at point (0,–4), the y-intercept.

Example 2: Using the point-slope form find the equation of a line passing through the point (3,2) and having a slope of 2.

$$\begin{aligned} (y - 2) &= 2(x - 3) \\ y &= 2x - 6 + 2 \\ y &= 2x - 4 \end{aligned}$$

Because the slope, 2, is positive the line is increasing and the line passes through the y-axis at the y-intercept at a value of –4.

Polar Coordinates.— Another coordinate system is determined by a fixed point O, the origin or pole, and a zero direction or axis through it, on which positive lengths can be laid off and measured, as a number line. A point P can be fixed to the zero direction line at a distance r away and then rotated in a positive sense at an angle u. The angle, u, in polar coordinates can take on values from 0° to 360°. A point in polar coordinates takes the form of (u, r).

Changing Coordinate Systems.—For simplicity it may be assumed that the origin on a Cartesian coordinate system coincides with the pole on a polar coordinate system, and it's axis with the x-axis. Then, if point P has polar coordinates of (u, r) and Cartesian coordinates of (x, y), by trigonometry $x = r \times \cos(u)$ and $y = r \times \sin(u)$. By the Pythagorean theorem and trigonometry

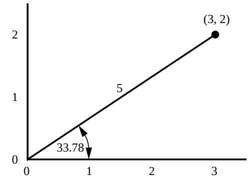
$$r = \sqrt{x^2 + y^2} \qquad \theta = \operatorname{atan}\frac{y}{x}$$

Example 1: Convert the Cartesian coordinate (3, 2) into polar coordinates.

$$r = \sqrt{3^2 + 2^2} = \sqrt{9 + 4} = \sqrt{13} = 3.6 \qquad \theta = \operatorname{atan}\frac{2}{3} = 33.78°$$

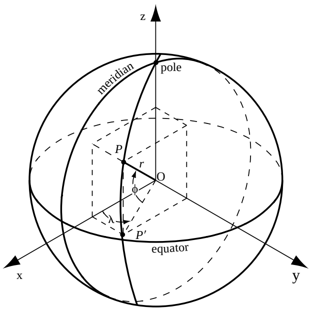
Therefore the point (3.6, 33.78) is the polar form of the Cartesian point (3, 2).

Graphically, the polar and Cartesian coordinates are related in the following figure



Example 2: Convert the polar form (5, 608) to Cartesian coordinates. By trigonometry, $x = r \times \cos(u)$ and $y = r \times \sin(u)$. Then $x = 5 \cos 608 = 2.5$ and $y = 5 \sin 608 = 4.33$. Therefore, the Cartesian point equivalent is (2.5, 4.33).

Spherical Coordinates.—It is convenient in certain problems, for example, those concerned with spherical surfaces, to introduce non-parallel coordinates. An arbitrary point *P* in space can be expressed in terms of the distance *r* between point *P* and the origin *O*, the angle ϕ that *OP′* makes with the *x–y* plane, and the angle λ that the projection *OP′* (of the segment *OP* onto the *x–y* plane) makes with the positive *x*-axis.



The rectangular coordinates of a point in space can therefore be calculated by the following formulas

Relationship Between Spherical and Rectangular Coordinates

Spherical to Rectangular	Rectangular to Spherical	
$x = \cos\phi \cos\lambda$	$r = \sqrt{x^2 + y^2 + z^2}$	
$y = r \cos\phi \sin\lambda$	$\phi = \operatorname{atan}\frac{z}{\sqrt{x^2 + y^2}}$	(for $x^2 + y^2 \neq 0$)
$z = r \sin\phi$	$\lambda = \operatorname{atan}\frac{y}{x}$	(for $x > 0, y > 0$)
	$\lambda = \pi + \operatorname{atan}\frac{y}{x}$	(for $x < 0$)
	$\lambda = 2\pi + \operatorname{atan}\frac{y}{x}$	(for $x > 0$, y < 0)

Example: What are the spherical coordinates of the point $P(3, 4, -12)$?

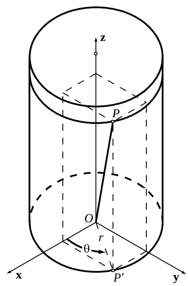
$$r = \sqrt{3^2 + (-4)^2 + (-12)^2} = 13$$

$$\phi = \operatorname{atan}\frac{-12}{\sqrt{3^2 + (-4)^2}} = \operatorname{atan}-\frac{12}{5} = -67.38°$$

$$\lambda = 360° + \operatorname{atan}-\frac{4}{3} = 360° - 53.13° = 306.87°$$

The spherical coordinates of P are therefore $r = 13$, $\phi = 267.388$, and $\lambda = 306.878$.

Cylindrical Coordinates.—For problems on the surface of a cylinder it is convenient to use cylindrical coordinates. The cylindrical coordinates r, θ, z, of P coincide with the polar coordinates of the point P′ in the x-y plane and the rectangular z-coordinate of P. This gives the conversion formula. Those for θ hold only if $x^2 + y^2 \neq 0$; θ is undetermined if $x = y = 0$.

Cylindrical to Rectangular	Rectangular to Cylindrical	
$x = r\cos\theta$	$r = \frac{1}{\sqrt{x^2 + y^2}}$	
$y = r\sin\theta$	$\cos\theta = \frac{x}{\sqrt{x^2 + y^2}}$ $\sin\theta = \frac{y}{\sqrt{x^2 + y^2}}$	
$z = z$	$z = z$	

Example: Given the cylindrical coordinates of a point P, $r = 3$, $\theta = -30°$, $z = 51$, find the rectangular coordinates. Using the above formulas $x = 3\cos(-30°) = 3\cos(30°) = 2.598$; $y = 3\sin(-30°) = -3\sin(30°) = -1.5$; and $z = 51$. Therefore, the rectangular coordinates of point P are $x = 2.598$, $y = -1.5$, and $z = 51$.

Imaginary and Complex Numbers

Complex or Imaginary Numbers.—Complex or imaginary numbers represent a class of mathematical objects that are used to simplify certain problems, such as the solution of polynomial equations. The basis of the complex number system is the unit imaginary number i that satisfies the following relations:

$$i^2 = (-i)^2 = -1 \qquad i = \sqrt{-1} \qquad -i = -\sqrt{-1}$$

In electrical engineering and other fields, the unit imaginary number is often represented by j rather than i. However, the meaning of the two terms is identical.

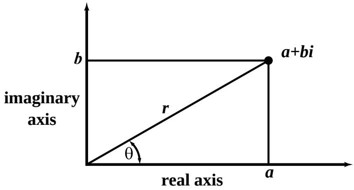
Rectangular or Trigonometric Form: Every complex number, Z, can be written as the sum of a real number and an imaginary number. When expressed as a sum, $Z = a + bi$, the complex number is said to be in rectangular or trigonometric form. The real part of the number is a, and the imaginary portion is bi because it has the imaginary unit assigned to it.

Polar Form: A complex number $Z = a + bi$ can also be expressed in polar form, also known as phasor form. In polar form, the complex number Z is represented by a magnitude r and an angle θ as follows:

$$Z = r\angle\theta$$

where $\angle\theta$ = a direction, the angle whose tangent is $b \div a$, thus $\theta = \operatorname{atan}\frac{b}{a}$; and, r = a magnitude = $\sqrt{a^2 + b^2}$.

A complex number can be plotted on a real-imaginary coordinate system known as the complex plane. The figure below illustrates the relationship between the rectangular coordinates *a* and *b,* and the polar coordinates *r* and θ.



Complex Number in the Complex Plane

The rectangular form can be determined from *r* and θ as follows:

$$a = r\cos\theta \qquad b = r\sin\theta \qquad a + bi = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$$

The rectangular form can also be written using Euler's Formula:

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Complex Conjugate: Complex numbers commonly arise in finding the solution of polynomials. A polynomial of n^{th} degree has *n* solutions, an even number of which are complex and the rest are real. The complex solutions always appear as complex conjugate pairs in the form $a + bi$ and $a - bi$. The product of these two conjugates, $(a + bi) \times (a - bi) = a^2 + b^2$, is the square of the magnitude *r* illustrated in the previous figure.

Operations on Complex Numbers

Example 3, Addition: When adding two complex numbers, the real parts and imaginary parts are added separately, the real parts added to real parts and the imaginary to imaginary parts. Thus,

$$(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$$

$$(a_1 + ib_1) - (a_2 + ib_2) = (a_1 - a_2) + i(b_1 - b_2)$$

$$(3 + 4i) + (2 + i) = (3 + 2) + (4 + 1)i = 5 + 5i$$

Example 4, Multiplication: Multiplication of two complex numbers requires the use of the imaginary unit, $i^2 = -1$ and the algebraic distributive law.

$$(a_1 + ib_1)(a_2 + ib_2) = a_1a_2 + ia_1b_2 + ia_2b_1 + i^2b_1b_2$$

$$= a_1a_2 + ia_1b_2 + ia_2b_1 - b_1b_2$$

$$(7 + 2i) \times (5 - 3i) = (7)(5) - (7)(3i) + (2i)(5) - (2i)(3i)$$

$$= 35 - 21i + 10i - 6i^2$$

$$= 35 - 21i + 10i - (6)(-1)$$

$$= 41 - 11i$$

Multiplication of two complex numbers, $Z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $Z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, results in the following:

$$Z_1 \times Z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2) = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

Example 5, Division: Divide the following two complex numbers, 2 + 3*i* and 4 – 5*i*. Dividing complex numbers makes use of the complex conjugate.

$$\frac{2+3i}{4-5i} = \frac{(2+3i)(4+5i)}{(4-5i)(4+5i)} = \frac{8+12i+10i+15i^2}{16+20i-20i-25i^2} = \frac{-7+22i}{16+25} = \left(\frac{-7}{41}\right)+i\left(\frac{22}{41}\right)$$

Example 6: Convert the complex number 8+6*i* into phasor form.

First find the magnitude of the phasor vector and then the direction.

$$\text{magnitude} = \sqrt{8^2+6^2} = \sqrt{100} = 10 \qquad \text{direction} = \text{atan}\frac{6}{8} = \text{atan}\,0.75 = 36.87°$$

$$\text{phasor} = 10\angle 36.87°$$

Break-Even Analysis

Break-Even Analysis.—Break-even analysis is a method of comparing two or more alternatives to determine which works best. Frequently, cost is the basis of the comparison, with the least expensive alternative being the most desirable. Break-even analysis can be applied in situations such as: to determine if it is more efficient and cost effective to use HSS, carbide, or ceramic tooling; to compare coated versus uncoated carbide tooling; to decide which of several machines should be used to produce a part; or to decide whether to buy a new machine for a particular job or to continue to use an older machine. The techniques used to solve any of these problems are the same; however, the details will be different, depending on the type of comparison being made. The remainder of this section deals with break-even analysis based on comparing the costs of manufacturing a product using different machines.

Choosing a Manufacturing Method: The object of this analysis is to decide which of several machines can produce parts at the lowest cost. In order to compare the cost of producing a part, all the costs involved in making that part must be considered. The cost of manufacturing any number of parts can be expressed as the sum: $C_T = C_F + n \times C_V$, where C_T is the total cost of manufacturing one part, C_F is the sum of the fixed costs of making the parts, n is the number of parts made, and C_V is the total variable costs per piece made.

Fixed costs are manufacturing costs that have to be paid whatever the number of parts is produced and usually before any parts can be produced. They include the cost of drafting and CNC part programs, the cost of special tools and equipment required to make the part, and the cost of setting up the machine for the job. Fixed costs are generally one-time charges that occur at the beginning of a job or are recurrent charges that do not depend on the number of pieces made, such as those that might occur each time a job is run again.

Variable costs depend on the number of parts produced and are expressed as the cost per part made. The variable costs include the cost of materials, the cost of machine time, the cost of the labor directly involved in making the part, and the portion of the overhead that is attributable to production of the part. Variable costs can be expressed as: C_V = material cost + machine cost + labor cost + overhead cost. When comparing alternatives, if the same cost is incurred by each alternative, then that cost can be eliminated from the analysis without affecting the result. For example, the cost of material is frequently omitted from a manufacturing analysis if each machine is going to make parts from the same stock and if there is not going to be a significant difference in the amount of scrap produced by each method. The time to produce one part is needed to determine the machine, labor, and overhead costs. The total time expressed in hours per part is $t_T = t_f + t_s$, where t_f equals the floor-to-floor production time for one part and t_s the setup time per part. The setup time, t_s, is the time spent setting up the machine and periodically reconditioning tooling, divided by the number of parts made per setup.

Material cost equals the cost of the materials divided by the number of parts made.

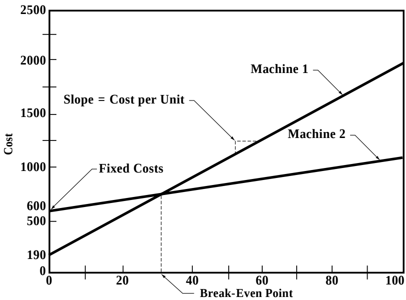
Machine cost is the portion of a machine's total cost that is charged toward the production of each part. It is found by multiplying the machine rate (cost of the machine per hour) by the machine time per part, t_f. The machine hourly rate is calculated by dividing the lifetime costs (including purchase price, insurance, maintenance, etc.) by the estimated lifetime hours of operation of the machine. The total operating hours may be difficult to determine but a reasonable number can be based on experience and dealer information.

Labor costs are the wages paid to people who are directly involved in the manufacture of the part. The labor cost per part is the labor rate per hour multiplied by the time needed to manufacture each part, t_T. Indirect labor, which supports but is not directly involved in the manufacture of the part, is charged as overhead.

Overhead cost is the cost of producing an item that is not directly related to the cost of manufacture. Overhead includes the cost of management and other support personnel, building costs, heating and cooling, and similar expenses. Often, overhead is estimated as a percentage of the largest component cost of producing a part. For example, if direct labor is the largest expense in producing a part, the overhead can be estimated as a percentage of the direct labor costs. On the other hand, if equipment costs are higher, the overhead would be based on a percentage of the machine cost. Depending on the company, typical overhead charges range from about 150 to 800 per cent of the highest variable cost.

Most of the time, the decision to use one machine or another for making parts depends on how many pieces are needed. For example, given three machines *A*, *B*, and *C*, if only a few parts need to be produced, then, in terms of cost, machine *A* might be the best; if hundreds of parts are needed, then machine *B* might be best; and, if thousands of components are to be manufactured, then machine *C* may result in the lowest cost per part. Break-even analysis reveals how many components need to be produced before a particular machine becomes more cost effective than another.

To use break-even analysis, the cost of operating each machine needs to be established. The costs are plotted on a graph as a function of the number of components to be manufactured to learn which machine can make the required parts for the least cost. The following graph is a plot of the fixed and variable costs of producing a quantity of parts on two different machines, *Machine 1* and *Machine 2*. Fixed costs for each machine are plotted on the vertical *cost* axis. Variable costs for each machine are plotted as a line that intersects the cost axis at the fixed cost for each respective machine. The variable cost line is constructed with a slope that is equal to the cost per part, that is, for each part made, the line rises by an amount equal to the cost per part. If the calculations necessary to produce the graph are done carefully, the total cost of producing any quantity of parts can be found from the data plotted on the graph.



Quantity of Parts

As an example, the graph shown is a comparison of the cost of manufacturing a quantity of a small part on a manually operated milling machine (*Machine 1*) and on a CNC machining center (*Machine 2*). The fixed costs (fixed costs = lead time × lead time rate + setup time × setup rate) for the manual machine are \$190 and the fixed costs for the CNC machine are higher at \$600. The fixed cost for each machine is the starting point of the line representing the cost of manufacturing a quantity of parts with that machine. The variable costs plotted are: \$18 per piece for the manual machine and \$5 per piece for the CNC mill.

The variable costs are calculated using the machine, labor, and overhead costs. The cost of materials is not included because it is assumed that materials cost will be the same for parts made on either machine and there will be no appreciable difference in the amount of scrap generated. The original cost of *Machine 1* (the manual milling machine) is \$19,000 with an estimated operating life of 16,000 hours, so the hourly operating cost is 19,000/16,000 = \$1.20 per hour. The labor rate is \$17 per hour and the overhead is estimated as 1.6 times the labor rate, or \$17 × 1.6 = \$27.20 per hour. The time, t_f, needed to complete each part on *Machine 1* is estimated as 24 minutes (0.4 hour). Therefore, by using *Machine 1*, the variable cost per part excluding material is (1.20 + 17.00 + 27.20) \$/h × 0.4 h/part = \$18 per part. For *Machine 2* (the CNC machining center), the machine cost is calculated at \$3 per hour, which is based on a \$60,000 initial cost (including installation, maintenance, insurance, etc.) and 20,000 hours of estimated lifetime. The cost of labor is \$15 per hour for *Machine 2* and the overhead is again calculated at 1.6 times the labor rate, or \$24 per hour. Each part is estimated to take 7.2 minutes (0.12 h) to make, so the variable cost per part made on *Machine 2* is (3 + 15 + 24) \$/h × 0.12 h/part = \$5 per part.

The lines representing the variable cost of operating each machine intersect at only one point on the graph. The intersection point corresponds to a quantity of parts that can be made by either the CNC or manual machine for the same cost, which is the break-even point. In the figure, the break-even point is 31.5 parts and the cost of those parts is \$757, or about \$24 apiece, excluding materials. The graph shows that if fewer than 32 parts need to be made, the total cost will be lowest if the manual machine is used because the line representing *Machine 1* is lower (representing lower cost) than the line representing *Machine 2*. On the other hand, if more than 31 parts are going to be made, the CNC machine will produce them for a lower cost. It is easy to see that the per piece cost of manufacturing is lower on the CNC machine because the line for *Machine 2* rises at a slower rate than the line for *Machine 1*. For producing only a few parts, the manual machine will make them less expensively than the CNC because the fixed costs are lower, but once the CNC part program has been written, the CNC can also run small batches efficiently because very little setup work is required.

The quantity of parts corresponding to the break-even point is known as the break-even quantity Q_b. The break-even quantity can be found without the use of the graph by using the following break-even equation: $Q_b = (C_{F1} - C_{F2})/(C_{V2} - C_{V1})$. In this equation, the C_{F1} and C_{F2} are the fixed costs for *Machine 1* and *Machine 2*, respectively: C_{V1} and C_{V2} are the variable costs for *Machine 1* and *Machine 2*, respectively.

Break-even analysis techniques are also useful for comparing performance of more than two machines. Plot the manufacturing costs for each machine on a graph as before and then compare the costs of the machines in pairs using the techniques described. For example, if an automatic machine such as a rotary transfer machine is included as *Machine 3* in the preceding analysis, then three lines representing the costs of operating each machine would be plotted on the graph. The equation to find the break-even quantities is applied three times in succession, for *Machines 1* and *2*, for *Machines 1* and *3*, and again for *Machines 2* and *3*. The result of this analysis will show the region (range of quantities) within which each machine is most profitable.

GEOMETRY

Arithmetical Progression

An arithmetical progression is a series of numbers in which each consecutive term differs from the preceding one by a fixed amount called the *common difference*, d. Thus, 1, 3, 5, 7, etc., is an arithmetical progression where the difference d is 2. The difference here is *added* to the preceding term, and the progression is called increasing. In the series 13, 10, 7, 4, etc., the difference is (–3), and the progression is called decreasing. In any arithmetical progression (or part of progression), let

a = first term considered
l = last term considered
n = number of terms
d = common difference
S = sum of n terms

Then the general formulas are $l = a + (n - 1)d$ and $S = \frac{a + l}{2} \times n$

In these formulas, d is positive in an increasing and negative in a decreasing progression. When any three of the preceding live quantities are given, the other two can be found by the formulas in the accompanying table of arithmetical progression.

Example: In an arithmetical progression, the first term equals 5, and the last term 40. The difference is 7. Find the sum of the progression.

$$S = \frac{a + l}{2d}(l + d - a) = \frac{5 + 40}{2 \times 7}(40 + 7 - 5) = 135$$

Geometrical Progression

A geometrical progression or a geometrical series is a series in which each term is derived by multiplying the preceding term by a constant multiplier called the *ratio*. When the ratio is greater than 1, the progression is increasing; when less than 1, it is decreasing. Thus, 2, 6, 18, 54, etc., is an increasing geometrical progression with a ratio of 3, and 24, 12, 6, etc., is a decreasing progression with a ratio of 1⁄2.

In any geometrical progression (or part of progression), let

a = first term
l = last (or nth) term
n = number of terms
r = ratio of the progression
S = sum of n terms

Then the general formulas are $l = ar^{n-1}$ and $S = \frac{rl - a}{r - 1}$

When any three of the preceding five quantities are given, the other two can be found by the formulas in the accompanying table. For instance, geometrical progressions are used for finding the successive speeds in machine tool drives, and in interest calculations.

Example: The lowest speed of a lathe is 20 rpm. The highest speed is 225 rpm. There are 18 speeds. Find the ratio between successive speeds.

$$\text{Ratio } r = \sqrt[n-1]{\frac{l}{a}} = \sqrt[17]{\frac{225}{20}} = \sqrt[17]{11.25} = 1.153$$

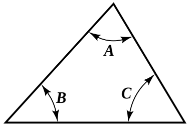
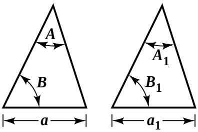
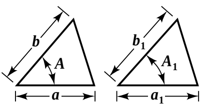
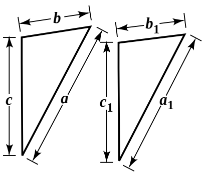
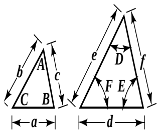
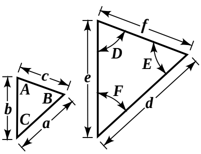
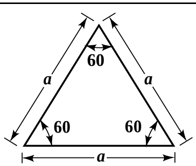
Formulas for Arithmetical Progression

To Find	Given	Use Equation
a	d l n	$a = l-(n-1)d$
	d n S	$a = \frac{S}{n} - \frac{n-1}{2} \times d$
	d l S	$a = \frac{d}{2} \pm \frac{1}{2}\sqrt{(2l+d)^2 - 8dS}$
	l n S	$a = \frac{2S}{n} - l$
d	a l n	$d = \frac{l-a}{n-1}$
	a n S	$d = \frac{2S-2an}{n(n-1)}$
	a l S	$d = \frac{l^2-a^2}{2S-l-a}$
	l n S	$d = \frac{2nl-2S}{n(n-1)}$
l	a d n	$l = a+(n-1)d$
	a d S	$l = -\frac{d}{2} \pm \frac{1}{2}\sqrt{8dS+(2a-d)^2}$
	a n S	$l = \frac{2S}{n} - a$
	d n S	$l = \frac{S}{n} + \frac{n-1}{2} \times d$
n	a d l	$n = 1 + \frac{l-a}{d}$
	a d S	$n = \frac{d-2a}{2d} \pm \frac{1}{2d}\sqrt{8dS+(2a-d)^2}$
	a l S	$n = \frac{2S}{a+l}$
	d l S	$n = \frac{2l+d}{2d} \pm \frac{1}{2d}\sqrt{(2l+d)^2-8dS}$
S	a d n	$S = \frac{n}{2}[2a+(n-1)d]$
	a d l	$S = \frac{a+l}{2} + \frac{l^2-a^2}{2d} = \frac{a+l}{2d}(l+d-a)$
	a l n	$S = \frac{n}{2}(a+l)$
	d l n	$S = \frac{n}{2}[2l-(n-1)d]$

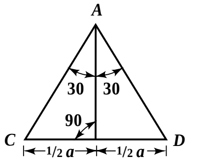
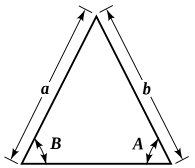
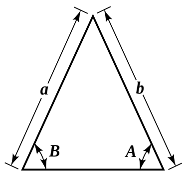
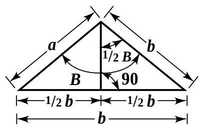
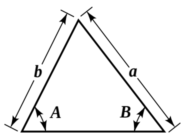
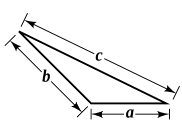
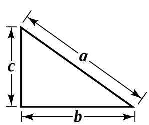
Formulas for Geometrical Progression

To Find	Given			Use Equation
a	l	n	r	$a = \frac{l}{r^{n-1}}$
	n	r	S	$a = \frac{(r-1)S}{r^n - 1}$
	l	r	S	$a = lr - (r-1)S$
	l	n	S	$a(S-a)^{n-1} = l(S-1)^{n-1}$
l	a	n	r	$l = ar^{n-1}$
	a	r	S	$l = \frac{1}{r}[a + (r-1)S]$
	a	n	S	$l(S-l)^{n-1} = a(S-a)^{n-1}$
	n	r	S	$l = \frac{S(r-1)r^{n-1}}{r^n - 1}$
n	a	l	r	$n = \frac{\log l - \log a}{\log r} + 1$
	a	r	S	$n = \frac{\log[a + (r-1)S] - \log a}{\log r}$
	a	l	S	$n = \frac{\log l - \log a}{\log(S-a) - \log(S-l)} + 1$
	l	r	S	$n = \frac{\log l - \log[lr - (r-1)S]}{\log r} + 1$
r	a	l	n	$r = \sqrt[n-1]{\frac{l}{a}}$
	a	n	S	$r^n = \frac{Sr}{a} + \frac{a-S}{a}$
	a	l	S	$r = \frac{S-a}{S-l}$
	l	n	S	$r^n = \frac{Sr^{n-1}}{S-l} - \frac{l}{S-l}$
S	a	n	r	$S = \frac{a(r^n - 1)}{r-1}$
	a	l	r	$S = \frac{lr - a}{r-1}$
	a	l	n	$S = \frac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}$
	l	n	r	$S = \frac{l(r^n - 1)}{(r-1)r^{n-1}}$

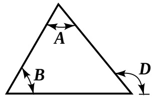
Geometrical Propositions

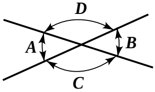
	The sum of the three angles in a triangle always equals 180 degrees. Hence, if two angles are known, the third angle can always be found. $A + B + C = 180°$ $A = 180° - (B + C)$ $B = 180° - (A + C)$ $C = 180° - (A + B)$
	If one side and two angles in one triangle are equal to one side and similarly located angles in another triangle, then the remaining two sides and angle also are equal. If $a = a_1$, $A = A_1$, and $B = B_1$, then the two other sides and the remaining angle also are equal.
	If two sides and the angle between them in one triangle are equal to two sides and a similarly located angle in another triangle, then the remaining side and angles also are equal. If $a = a_1$, $b = b_1$, and $A = A_1$, then the remaining side and angles also are equal.
	If the three sides in one triangle are equal to the three sides of another triangle, then the angles in the two triangles also are equal. If $a = a_1$, $b = b_1$, and $c = c_1$, then the angles between the respective sides also are equal.
	If the three sides of one triangle are proportional to corresponding sides in another triangle, then the triangles are called *similar*, and the angles in the one are equal to the angles in the other. If $a : b : c = d : e : f$, then $A = D$, $B = E$, and $C = F$.
	If the angles in one triangle are equal to the angles in another triangle, then the triangles are similar and their corresponding sides are proportional. If $A = D, B = E$, and $C = F$, then $a : b : c = d : e : f$.
	If the three sides in a triangle are equal—that is, if the triangle is *equilateral*—then the three angles also are equal. Each of the three equal angles in an equilateral triangle is 60 degrees. If the three angles in a triangle are equal, then the three sides also are equal.

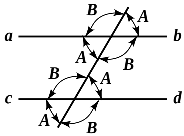
Geometrical Propositions

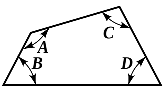
	A line in an equilateral triangle that bisects or divides any of the angles into two equal parts also bisects the side opposite the angle and is at right angles to it. If line *AB* divides angle *CAD* into two equal parts, it also divides line *CD* into two equal parts and is at right angles to it.
	If two sides in a triangle are equal—that is, if the triangle is an *isosceles* triangle—then the angles opposite these sides also are equal. If side *a* equals side *b*, then angle *A* equals angle *B*.
	If two angles in a triangle are equal, the sides opposite these angles also are equal. If angles *A* and *B* are equal, then side *a* equals side *b*.
	In an isosceles triangle, if a straight line is drawn from the point where the two equal sides meet, so that it bisects the third side or base of the triangle, then it also bisects the angle between the equal sides and is perpendicular to the base.
	In every triangle, that angle is greater that is opposite a longer side. In every triangle, that side is greater which is opposite a greater angle. If *a* is longer than *b*, then angle *A* is greater than *B*. If angle *A* is greater than *B*, then side *a* is longer than *b*.
	In every triangle, the sum of the lengths of two sides is always greater than the length of the third. Side *a* + side *b* is always greater than side *c*.
	In a right-angle triangle, the square of the hypotenuse or the side opposite the right angle is equal to the sum of the squares on the two sides that form the right angle. $a^2 = b^2 + c^2$

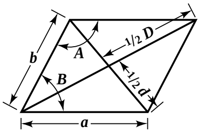
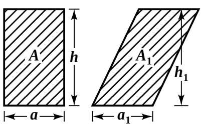
Geometrical Propositions

<table>
<tr><td>
</td><td>If one side of a triangle is produced, then the exterior angle is equal to the sum of the two interior opposite angles.

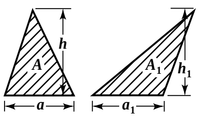
$$\text{Angle } D = \text{angle } A + \text{angle } B$$</td></tr>
<tr><td>
</td><td>If two lines intersect, then the opposite angles formed by the intersecting lines are equal.

$$\text{Angle } A = \text{angle } B$$
$$\text{Angle } C = \text{angle } D$$</td></tr>
<tr><td>
</td><td>If a line intersects two parallel lines, then the corresponding angles formed by the intersecting line and the parallel lines are equal.

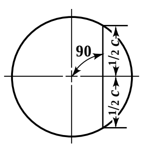
Lines *ab* and *cd* are parallel. Then all the angles designated *A* are equal, and all those designated *B* are equal.</td></tr>
<tr><td>
</td><td>In any figure having four sides, the sum of the interior angles equals 360 degrees.

$$A + B + C + D = 360 \text{ degrees}$$</td></tr>
<tr><td>
</td><td>The sides that are opposite each other in a parallelogram are equal; the angles that are opposite each other are equal; the diagonal divides it into two equal parts. If two diagonals are drawn, they bisect each other.</td></tr>
<tr><td>
</td><td>The areas of two parallelograms that have equal base and equal height are equal.

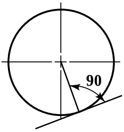
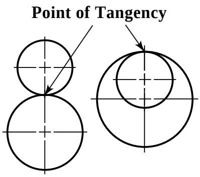
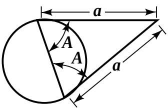
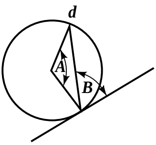
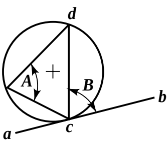
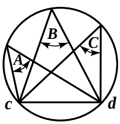
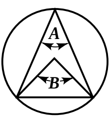
If $a = a_1$ and $h = h_1$, then

$$\text{Area } A = \text{area } A_1$$</td></tr>
<tr><td>
</td><td>The areas of triangles having equal base and equal height are equal.

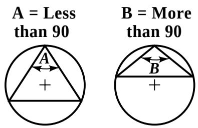
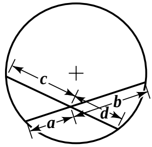
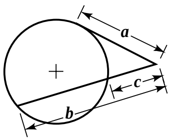
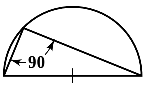
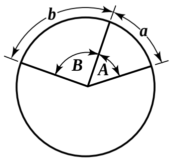
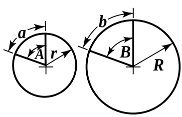
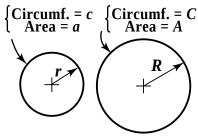
If $a = a_1$ and $h = h_1$, then

$$\text{Area } A = \text{area } A_1$$</td></tr>
<tr><td>
</td><td>If a diameter of a circle is at right angles to a chord, then it bisects or divides the chord into two equal parts.</td></tr>
</table>

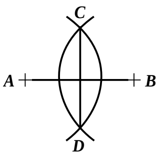
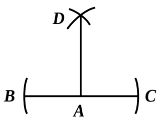
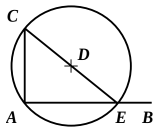
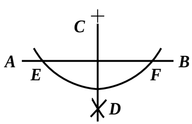
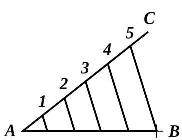
Geometrical Propositions

	If a line is tangent to a circle, then it is also at right angles to a line drawn from the center of the circle to the point of tangency—that is, to a radial line through the point of tangency.
	If two circles are tangent to each other, then the straight line that passes through the centers of the two circles must also pass through the point of tangency.
	If from a point outside a circle, tangents are drawn to a circle, the two tangents are equal and make equal angles with the chord joining the points of tangency.
	The angle between a tangent and a chord drawn from the point of tangency equals one-half the angle at the center subtended by the chord. Angle $B = \frac{1}{2}$ angle A
	The angle between a tangent and a chord drawn from the point of tangency equals the angle at the periphery subtended by the chord. Angle *B*, between tangent *ab* and chord *cd*, equals angle *A* subtended at the periphery by chord *cd*.
	All angles having their vertex at the periphery of a circle and subtended by the same chord are equal. Angles *A*, *B*, and *C*, all subtended by chord *cd*, are equal.
	If an angle at the circumference of a circle, between two chords, is subtended by the same arc as the angle at the center, between two radii, then the angle at the circumference is equal to one-half of the angle at the center. Angle $A = \frac{1}{2}$ angle B

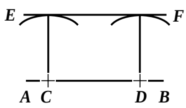
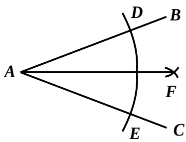
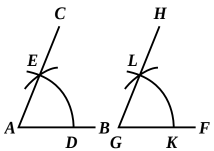
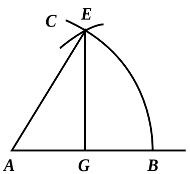
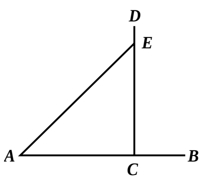
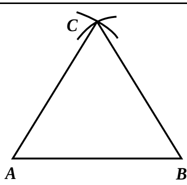
Geometrical Propositions

	An angle subtended by a chord in a circular segment larger than one-half the circle is an acute angle—an angle less than 90 degrees. An angle subtended by a chord in a circular segment less than one-half the circle is an obtuse angle—an angle greater than 90 degrees.
	If two chords intersect each other in a circle, then the rectangle of the segments of the one equals the rectangle of the segments of the other. $$a \times b = c \times d$$
	If from a point outside a circle two lines are drawn, one of which intersects the circle and the other is tangent to it, then the rectangle contained by the total length of the intersecting line, and that part of it that is between the outside point and the periphery, equals the square of the tangent. $$a^2 = b \times c$$
	If a triangle is inscribed in a semicircle, the angle opposite the diameter is a right (90-degree) angle. All angles at the periphery of a circle, subtended by the diameter, are right (90-degree) angles.
	The lengths of circular arcs of the same circle are proportional to the corresponding angles at the center. $$A : B = a : b$$
	The lengths of circular arcs having the same center angle are proportional to the lengths of the radii. If $A = B$, then $a : b = r : R$.
	The circumferences of two circles are proportional to their radii. The areas of two circles are proportional to the squares of their radii. $$c : C = r : R$$ $$a : A = r^2 : R^2$$

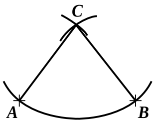
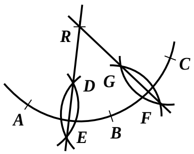
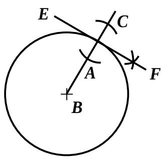
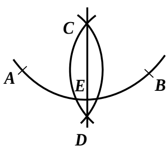
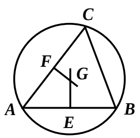
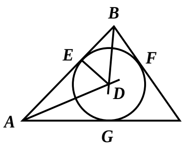
Geometrical Constructions

	To divide a line *AB* into two equal parts: With the ends *A* and *B* as centers and a radius greater than one-half the line, draw circular arcs. Through the intersections *C* and *D*, draw line *CD*. This line divides *AB* into two equal parts and is also perpendicular to *AB*.
	To draw a perpendicular to a straight line from a point *A* on that line: With *A* as a center and with any radius, draw circular arcs intersecting the given line at *B* and *C*. Then, with *B* and *C* as centers and a radius longer than *AB*, draw circular arcs intersecting at *D*. Line *DA* is perpendicular to *BC* at *A*.
	To draw a perpendicular line from a point *A* at the end of a line *AB*: With any point *D*, outside of the line *AB*, as a center, and with *AD* as a radius, draw a circular arc intersecting *AB* at *E*. Draw a line through *E* and *D* intersecting the arc at *C*; then join *AC*. This line is the required perpendicular.
	To draw a perpendicular to a line *AB* from a point *C* at a distance from it: With *C* as a center, draw a circular arc intersecting the given line at *E* and *F*. With *E* and *F* as centers, draw circular arcs with a radius longer than one-half the distance between *E* and *F*. These arcs intersect at *D*. Line *CD* is the required perpendicular.
	To divide a straight line *AB* into a number of equal parts: Let it be required to divide *AB* into five equal parts. Draw line *AC* at an angle with *AB*. Set off on *AC* five equal parts of any convenient length. Draw *B*–5 and then draw lines parallel with *B*–5 through the other division points on *AC*. The points where these lines intersect *AB* are the required division points.

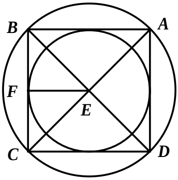
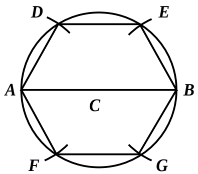
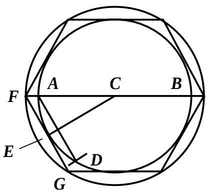
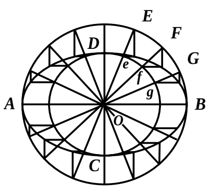
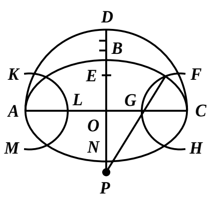
Geometrical Constructions

	To draw a straight line parallel to a given line *AB*, at a given distance from it: With any points *C* and *D* on *AB* as centers, draw circular arcs with the given distance as radius. Line *EF*, drawn to touch the circular arcs, is the required parallel line.
	To bisect or divide an angle *BAC* into two equal parts: With *A* as a center and any radius, draw arc *DE*. With *D* and *E* as centers and a radius greater than one-half *DE*, draw circular arcs intersecting at *F*. Line *AF* divides the angle into two equal parts.
	To draw an angle upon a line *AB*, equal to a given angle *FGH*: With point *G* as a center and with any radius, draw arc *KL*. With *A* as a center and with the same radius, draw arc *DE*. Make arc *DE* equal to *KL* and draw *AC* through *E*. Angle *BAC* then equals angle *FGH*.
	To lay out a 60-degree angle: With *A* as a center and any radius, draw an arc *BC*. With point *B* as a center and *AB* as a radius, draw an arc intersecting at *E* the arc just drawn. *EAB* is a 60-degree angle. A 30-degree angle may be obtained either by dividing a 60-degree angle into two equal parts or by drawing a line *EG* perpendicular to *AB*. Angle *AEG* is then 30 degrees.
	To draw a 45-degree angle: From point *A* on line *AB*, set off a distance *AC*. Draw the perpendicular *DC* and set off a distance *CE* equal to *AC*. Draw *AE*. Angle *EAC* is a 45-degree angle.
	To draw an equilateral triangle, the length of the sides of which equals *AB*: With *A* and *B* as centers and *AB* as radius, draw circular arcs intersecting at *C*. Draw *AC* and *BC*. Then *ABC* is an equilateral triangle.

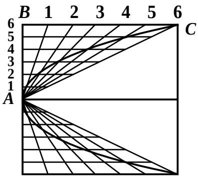
Geometrical Constructions

	To draw a circular arc with a given radius through two given points *A* and *B*: With *A* and *B* as centers, and the given radius as radius, draw circular arcs intersecting at *C*. With *C* as a center, and the same radius, draw a circular arc through *A* and *B*.
	To find the center of a circle or of an arc of a circle: Select three points on the periphery of the circle, as *A*, *B*, and *C*. With each of these points as a center and the same radius, describe arcs intersecting each other. Through the points of intersection, draw lines *DE* and *FG*. Point *H*, where these lines intersect, is the center of the circle.
	To draw a tangent to a circle from a given point on the circumference: Through the point of tangency *A*, draw a radial line *BC*. At point *A*, draw a line *EF* at right angles to *BC*. This line is the required tangent.
	To divide a circular arc *AB* into two equal parts: With *A* and *B* as centers, and a radius larger than half the distance between *A* and *B*, draw circular arcs intersecting at *C* and *D*. Line *CD* divides arc *AB* into two equal parts at *E*.
	To describe a circle about a triangle: Divide the sides *AB* and *AC* into two equal parts, and from the division points *E* and *F*, draw lines at right angles to the sides. These lines intersect at *G*. With *G* as a center and *GA* as a radius, draw circle *ABC*.
	To inscribe a circle in a triangle: Bisect two of the angles, *A* and *B*, by lines intersecting at *D*. From *D*, draw a line *DE* perpendicular to one of the sides, and with *DE* as a radius, draw circle *EFG*.

Geometrical Constructions

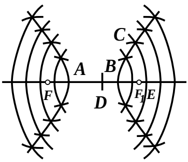
	To describe a circle about a square and to inscribe a circle in a square: The centers of both the circumscribed and inscribed circles are located at the point *E*, where the two diagonals of the square intersect. The radius of the circumscribed circle is *AE*, and of the inscribed circle, *EF*.
	To inscribe a hexagon in a circle: Draw a diameter *AB*. With *A* and *B* as centers and with the radius of the circle as radius, describe circular arcs intersecting the given circle at *D, E, F,* and *G*. Draw lines *AD, DE,* etc., forming the required hexagon.
	To describe a hexagon about a circle: Draw a diameter *AB*, and with *A* as a center and the radius of the circle as radius, cut the circumference of the given circle at *D*. Join *AD* and bisect it with radius *CE*. Through *E*, draw *FG* parallel to *AD* and intersecting line *AB* at *F*. With *C* as a center and *CF* as radius, draw a circle. Within this circle, inscribe the hexagon as in the preceding problem.
	To describe an ellipse with the given axes *AB* and *CD*: Describe circles with *O* as a center and *AB* and *CD* as diameters. From a number of points, *E, F, G*, etc., on the outer circle, draw radii intersecting the inner circle at *e, f,* and *g*. From *E, F,* and *G,* draw lines perpendicular to *AB*, and from *e, f,* and *g*, draw lines parallel to *AB*. The intersections of these perpendicular and parallel lines are points on the curve of the ellipse.
	To construct an approximate ellipse by circular arcs: Let *AC* be the major axis and *BN* the minor. Draw half circle *ADC* with *O* as a center. Divide *BD* into three equal parts and set off *BE* equal to one of these parts. With *A* and *C* as centers and *OE* as radius, describe circular arcs *KLM* and *FGH*; with *G* and *L* as centers, and the same radius, describe arcs *FCH* and *KAM*. Through *F* and *G*, drawn line *FP*, and with *P* as a center, draw the arc *FBK*. Arc *HNM* is drawn in the same manner.

Geometrical Constructions



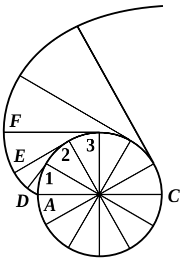
To construct a parabola:

Divide line *AB* into a number of equal parts and divide *BC* into the same number of parts. From the division points on *AB*, draw horizontal lines. From the division points on *BC*, draw lines to point *A*. The points of intersection between lines drawn from points numbered alike are points on the parabola.



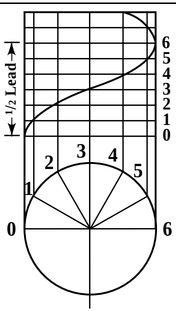
To construct a hyperbola:

From focus *F*, lay off a distance *FD* equal to the transverse axis, or the distance *AB* between the two branches of the curve. With *F* as a center and any distance *FE* greater than *FB* as a radius, describe a circular arc. Then with F_1 as a center and *DE* as a radius, describe arcs intersecting at *C* and *G* the arc just described. *C* and *G* are points on the hyperbola. Any number of points can be found in a similar manner.



To construct an involute:

Divide the circumference of the base circle *ABC* into a number of equal parts. Through the division points 1, 2, 3, etc., draw tangents to the circle and make the lengths *D*–1, *E*–2, *F*–3, etc., of these tangents equal to the actual length of the arcs *A*–1, *A*–2, *A*–3, etc.



To construct a helix:

Divide half the circumference of the cylinder, on the surface of which the helix is to be described, into a number of equal parts. Divide half the lead of the helix into the same number of equal parts. From the division points on the circle representing the cylinder, draw vertical lines, and from the division points on the lead, draw horizontal lines as shown. The intersections between lines numbered alike are points on the helix.

Areas and Volumes

The Prismoidal Formula.—The prismoidal formula is a general formula by which the volume of any prism, pyramid or frustum of a pyramid may be found.

A_1 = area at one end of the body

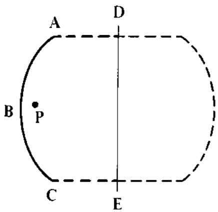
A_2 = area at the other end

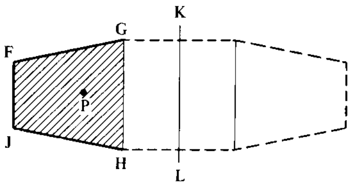
A_m = area of middle section between the two end surfaces

h = height of body

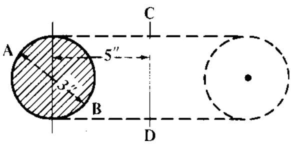
Then, volume V of the body is $V = \frac{h}{6}(A_1 + 4A_m + A_2)$

Pappus or Guldinus Rules.—By means of these rules the area of any surface of revolution and the volume of any solid of revolution may be found. The area of the surface swept out by the revolution of a line *ABC* (see illustration) about the axis *DE* equals the length of the line multiplied by the length of the path of its center of gravity, *P*. If the line is of such a shape that it is difficult to determine its center of gravity, then the line may be divided into a number of short sections, each of which may be considered as a straight line, and the areas swept out by these different sections, as computed by the rule given, may be added to find the total area. The line must lie wholly on one side of the axis of revolution and must be in the same plane.





The volume of a solid body formed by the revolution of a surface *FGHJ* about axis *KL* equals the area of the surface multiplied by the length of the path of its center of gravity. The surface must lie wholly on one side of the axis of revolution and in the same plane.

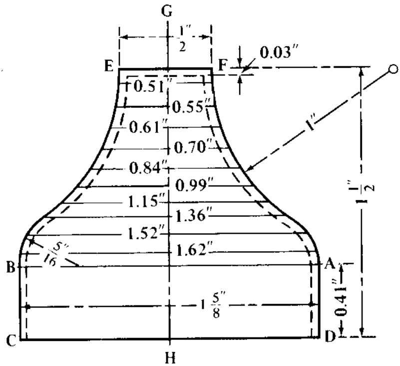


Example: By means of these rules, the area and volume of a cylindrical ring or torus may be found. The torus is formed by a circle *AB* being rotated about axis *CD*. The center of gravity of the circle is at its center. Hence, with the dimensions given in the illustration, the length of the path of the center of gravity of the circle is $3.1416 \times 10 = 31.416$ inches. Multiplying by the length of the circumference of the circle, which is $3.1416 \times 3 = 9.4248$ inches, gives $31.416 \times 9.4248 = 296.089$ square inches which is the area of the torus.

The volume equals the area of the circle, which is $0.7854 \times 9 = 7.0686$ square inches, multiplied by the path of the center of gravity, which is 31.416, as before; hence,

$$\text{Volume} = 7.0686 \times 31.416 = 222.067 \text{ cubic inches}$$

Approximate Method for Finding the Area of a Surface of Revolution.—The accompanying illustration is shown in order to give an example of the approximate method based on Guldinus' rule, that can be used for finding the area of a symmetrical body. In the illustration, the dimensions in common fractions are the known dimensions; those in decimals are found by actual measurements on a figure drawn to scale.



The method for finding the area is as follows: First, separate such areas as are cylindrical, conical, or spherical, as these can be found by exact formulas. In the illustration *ABCD* is a cylinder, the area of the surface of which can be easily found. The top area *EF* is simply a circular area, and can thus be computed separately. The remainder of the surface generated by rotating line *AF* about the axis *GH* is found by the approximate method explained in the previous section. From point *A*, set off equal distances on line *AF*. In the illustration, each division indicated is ⅛ inch long. From the central or middle point of each of these parts draw a line at right angles to the axis of rotation *GH*, measure the length of these lines or diameters (the length of each is given in decimals), add all these lengths together and multiply the sum by the length of one division set off on line *AF* (in this case, ⅛ inch), and multiply this product by π to find the approximate area of the surface of revolution.

In setting off divisions ⅛ inch long along line *AF*, the last division does not reach exactly to point *F*, but only to a point 0.03 inch below it. The part 0.03 inch high at the top of the cup can be considered as a cylinder of ½ inch diameter and 0.03 inch height, the area of the cylindrical surface of which is easily computed. By adding the various surfaces together, the total surface of the cup is found as follows:

Cylinder, 1 ⅝ inch diameter, 0.41 inch high	2.093 square inches
Circle, ½ inch diameter	0.196 square inch
Cylinder, ½ inch diameter, 0.03 inch high	0.047 square inch
Irregular surface	3.868 square inches
Total	6.204 square inches

Area of Plane Surfaces of Irregular Outline.—One of the most useful and accurate methods for determining the approximate area of a plane figure or irregular outline is known as *Simpson's Rule*. In applying Simpson's Rule to find an area the work is done in four steps:

1) Divide the area into an *even* number, N, of parallel strips of equal width W; for example, in the accompanying diagram, the area has been divided into 8 strips of equal width.

2) Label the sides of the strips V_0, V_1, V_2, etc., up to V_N.

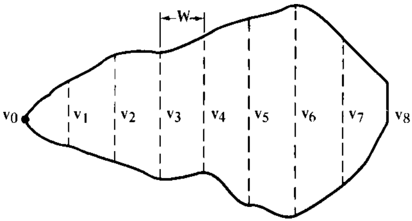
3) Measure the heights V_0, V_1, V_2, … , V_N of the sides of the strips.

4) Substitute the heights V_0, V_1, etc., in the following formula to find the area A of the figure:

$$A = \frac{W}{3}[(V_0 + V_N) + 4(V_1 + V_3 + \cdots + V_{N-1}) + 2(V_2 + V_4 + \cdots + V_{N \cdot}$$

Example: The area of the accompanying figure was divided into 8 strips on a full-size drawing and the following data obtained. Calculate the area using Simpson's Rule.

$W = \frac{1}{2}''$
$V_0 = 0''$
$V_1 = \frac{3}{4}''$
$V_2 = 1\frac{1}{4}''$
$V_3 = 1\frac{1}{2}''$
$V_4 = 1\frac{5}{8}''$
$V_5 = 2\frac{1}{4}''$
$V_6 = 2\frac{1}{2}''$
$V_7 = 1\frac{3}{4}''$
$V_8 = \frac{1}{2}''$



Substituting the given data in the Simpson formula,

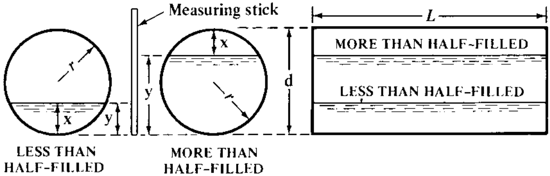
$$A = \frac{\frac{1}{2}}{3}[(0 + \tfrac{1}{2}) + 4(\tfrac{3}{4} + 1\tfrac{1}{2} + 2\tfrac{1}{4} + 1\tfrac{3}{4}) + 2(1\tfrac{1}{4} + 1\tfrac{5}{8} + 2\tfrac{1}{2})]$$
$$= \tfrac{1}{6}[(\tfrac{1}{2}) + 4(6\tfrac{1}{4}) + 2(5\tfrac{3}{8})] = \tfrac{1}{6}[36\tfrac{1}{4}]$$
$$= 6.04 \text{ square inches}$$

In applying Simpson's Rule, it should be noted that the larger the number of strips into which the area is divided the more accurate the results obtained.

Areas Enclosed by Cycloidal Curves.—The area between a cycloid and the straight line upon which the generating circle rolls, equals three times the area of the generating circle (see diagram, page 63). The areas between epicycloidal and hypocycloidal curves and the "fixed circle" upon which the generating circle is rolled, may be determined by the following formulas, in which a = radius of the fixed circle upon which the generating circle rolls; b = radius of the generating circle; A = the area for the epicycloidal curve; and A_1 = the area for the hypocycloidal curve.

$$A = \frac{3.1416b^2(3a + 2b)}{a} \qquad A_1 = \frac{3.1416b^2(3a - 2b)}{a}$$

Find the Contents of Cylindrical Tanks at Different Levels.—In conjunction with the table *Segments of Circles for Radius = 1* presented on pages 80 and 81, the following relations can give a close approximation of the liquid contents, at any level, in a cylindrical tank.



A long measuring rule calibrated in length units or simply a plain stick can be used for measuring contents at a particular level. In turn, the rule or stick can be graduated to serve as a volume gauge for the tank in question. The only requirements are that the cross-section of the tank is circular; the tank's dimensions are known; the gauge rod is inserted vertically through the top center of the tank so that it rests on the exact bottom of the tank; and that consistent English *or* metric units are used throughout the calculations.

$K = Cr^2L$ = Tank Constant (remains the same for any given tank) (1)

$V_T = \pi K$, for a tank that is completely full (2)

$V_s = KA$ (3)

$V = V_s$ when tank is less than half full (4)

$V = V_T - V_s = V_T - KA$, when tank is more than half full (5)

where C = liquid volume conversion factor, the exact value of which depends on the length and liquid volume units being used during measurement: 0.00433 U.S. gal/in³; 7.48 U.S. gal/ft³; 0.00360 U.K. gal/in³; 6.23 U.K. gal/ft³; 0.001 liter/cm³; or 1000 liters/m³

V_T = total volume of liquid tank can hold

V_s = volume formed by segment of circle having depth = x in given tank (see diagram)

V = volume of liquid at particular level in tank

d = diameter of tank; L = length of tank; r = radius of tank (= ½ diameter)

A = segment area of a corresponding unit circle taken from pages 80 or 81

y = actual depth of contents in tank as shown on a gauge rod or stick

x = depth of the segment of a circle to be considered in given tank. As can be seen in above diagram, x is the actual depth of contents (y) when the tank is less than half full, and is the depth of the void ($d - y$) above the contents when the tank is more than half full. From pages 80 and 81 it can also be seen that h, the height of a segment of a corresponding unit circle, is x/r

Example: A tank is 20 feet long and 6 feet in diameter. Convert a long inch-stick into a gauge that is graduated at 1000 and 3000 U.S. gallons.

$$L = 20 \times 12 = 240\text{in.} \qquad r = \tfrac{6}{2} \times 12 = 36\text{in.}$$

From Formula (1): $K = 0.00433(36)^2(240) = 1347$

From Formula (2): $V_T = 3.142 \times 1347 = 4232$ US gal.

The 72-inch mark from the bottom on the inch-stick can be graduated for the rounded full volume "4230"; and the halfway point 36″ for 4230⁄2 or "2115." It can be seen that the 1000-gal mark would be below the halfway mark. From Formulas (3) and (4):

$A_{1000} = \frac{1000}{1347} = 0.7424$ from page 81, h can be interpolated as 0.5724; and $x = y = 36 \times 0.5724 = 20.61$. If the desired level of accuracy permits, interpolation can be omitted by choosing h directly from the table on page 81 for the value of A nearest that calculated above.

Therefore, the 1000-gal mark is graduated 20⅝″ from bottom of rod.

It can be seen that the 3000 mark would be above the halfway mark. Therefore, the circular segment considered is the cross-section of the void space at the top of the tank. From Formulas (3) and (5):

$$A_{3000} = \frac{4230 - 3000}{1347} = 0.9131;\ h = 0.6648;\ x = 36 \times 0.6648 = 23.93''$$

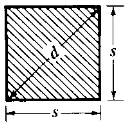
Therefore, the 3000-gal mark is 72.00 – 23.93 = 48.07, or at the 48 $\tfrac{1}{16}''$ mark from the bottom.

Areas and Dimensions of Plane Figures

In the following tables are given formulas for the areas of plane figures, together with other formulas relating to their dimensions and properties; the surfaces of solids; and the volumes of solids. The notation used in the formulas is, as far as possible, given in the illustration accompanying them; where this has not been possible, it is given at the beginning of each set of formulas.

Examples are given with each entry, some in English and some in metric units, showing the use of the preceding formula.

Square:



$$\text{Area} = A = s^2 = \tfrac{1}{2}d^2$$
$$s = 0.7071d = \sqrt{A}$$
$$d = 1.414s = 1.414\sqrt{A}$$

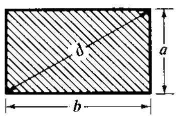
Example: Assume that the side s of a square is 15 inches. Find the area and the length of the diagonal.

$$\text{Area} = A = s^2 = 15^2 = 225 \text{ square inches}$$
$$\text{Diagonal} = d = 1.414s = 1.414 \times 15 = 21.21 \text{ inches}$$

Example: The area of a square is 625 square inches. Find the length of the side s and the diagonal d.

$$s = \sqrt{A} = \sqrt{625} = 25 \text{ inches}$$
$$d = 1.414\sqrt{A} = 1.414 \times 25 = 35.35 \text{ inches}$$

Rectangle:



$$\text{Area} = A = ab = a\sqrt{d^2 - a^2} = b\sqrt{d^2 - b^2}$$
$$d = \sqrt{a^2 + b^2}$$
$$a = \sqrt{d^2 - b^2} = A \div b$$
$$a = \sqrt{d^2 - a^2} = A \div a$$

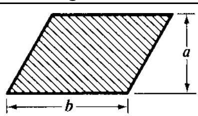
Example: The side a of a rectangle is 12 centimeters, and the area 70.5 square centimeters. Find the length of the side b, and the diagonal d.

$$b = A \div a = 70.5 \div 12 = 5.875 \text{ centimeters}$$
$$d = \sqrt{a^2 + b^2} = \sqrt{12^2 + 5.875^2} = \sqrt{178.516} = 13.361 \text{ centimeters}$$

Example: The sides of a rectangle are 30.5 and 11 centimeters long. Find the area.

$$\text{Area} = A = a \times b = 30.5 \times 11 = 335.5 \text{ square centimeters}$$

Parallelogram:



$$\text{Area} = A = ab$$
$$a = A \div b$$
$$b = A \div a$$

Note: The dimension a is measured at right angles to line b.

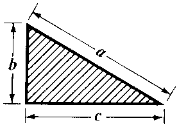
Example: The base b of a parallelogram is 16 feet. The height a is 5.5 feet. Find the area.

$$\text{Area} = A = a \times b = 5.5 \times 16 = 88 \text{ square feet}$$

Example: The area of a parallelogram is 12 square inches. The height is 1.5 inches. Find the length of the base b.

$$b = A \div a = 12 \div 1.5 = 8 \text{ inches}$$

Right-Angled Triangle:



$$\text{Area} = A = \frac{bc}{2}$$

$$a = \sqrt{b^2 + c^2}$$

$$b = \sqrt{a^2 - c^2}$$

$$c = \sqrt{a^2 - b^2}$$

Example: The sides b and c in a right-angled triangle are 6 and 8 inches. Find side a and the area

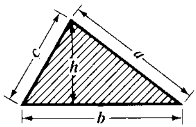
$$a = \sqrt{b^2 + c^2} = \sqrt{6^2 + 8^2} = \sqrt{36 + 64} = \sqrt{100} = 10 \text{ inches}$$

$$A = \frac{b \times c}{2} = \frac{6 \times 8}{2} = \frac{48}{2} = 24 \text{ square inches}$$

Example: If $a = 10$ and $b = 6$ had been known, but not c, the latter would have been found as follows:

$$c = \sqrt{a^2 - b^2} = \sqrt{10^2 - 6^2} = \sqrt{100 - 36} = \sqrt{64} = 8 \text{ inches}$$

Acute-Angled Triangle:



$$\text{Area} = A = \frac{bh}{2} = \frac{b}{2}\sqrt{a^2 - \left(\frac{a^2 + b^2 - c^2}{2b}\right)^2}$$

$$\text{If } S = \tfrac{1}{2}(a + b + c), \text{ then}$$

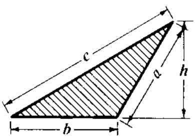
$$A = \sqrt{S(S-a)(S-b)(S-c)}$$

Example: If $a = 10$, $b = 9$, and $c = 8$ centimeters, what is the area of the triangle?

$$A = \frac{b}{2}\sqrt{a^2 - \left(\frac{a^2 + b^2 - c^2}{2b}\right)^2} = \frac{9}{2}\sqrt{10^2 - \left(\frac{10^2 + 9^2 - 8^2}{2 \times 9}\right)^2} = 4.5\sqrt{100 - \left(\frac{117}{18}\right)^2}$$

$$= 4.5\sqrt{100 - 42.25} = 4.5\sqrt{57.75} = 4.5 \times 7.60 = 34.20 \text{ square centimeters}$$

Obtuse-Angled Triangle:



$$\text{Area} = A = \frac{bh}{2} = \frac{b}{2}\sqrt{a^2 - \left(\frac{c^2 - a^2 - b^2}{2b}\right)^2}$$

$$\text{If S} = \tfrac{1}{2}(a + b + c), \text{ then}$$

$$A = \sqrt{S(S-a)(S-b)(S-c)}$$

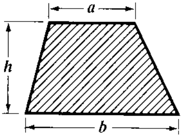
Example: The side $a = 5$, side $b = 4$, and side $c = 8$ inches. Find the area.

$$S = \tfrac{1}{2}(a + b + c) = \tfrac{1}{2}(5 + 4 + 8) = \tfrac{1}{2} \times 17 = 8.5$$

$$A = \sqrt{S(S-a)(S-b)(S-c)} = \sqrt{8.5(8.5-5)(8.5-4)(8.5-8)}$$

$$= \sqrt{8.5 \times 3.5 \times 4.5 \times 0.5} = \sqrt{66.937} = 8.18 \text{ square inches}$$

Trapezoid:



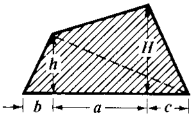
$$\text{Area} = A = \frac{(a + b)h}{2}$$

Note: In Britain, this figure is called a *trapezium* and the one below it is known as a *trapezoid*, the terms being reversed.

Example: Side $a = 23$ meters, side $b = 32$ meters, and height $h = 12$ meters. Find the area.

$$A = \frac{(a + b)h}{2} = \frac{(23 + 32)12}{2} = \frac{55 \times 12}{2} = 330 \text{ square meters}$$

Trapezium:



$$\text{Area} = A = \frac{(H+h)a + bh + cH}{2}$$

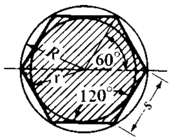
A trapezium can also be divided into two triangles as indicated by the dashed line. The area of each of these triangles is computed, and the results added to find the area of the trapezium.

Example: Let $a = 10$, $b = 2$, $c = 3$, $h = 8$, and $H = 12$ inches. Find the area.

$$A = \frac{(H+h)a + bh + cH}{2} = \frac{(12+8)10 + 2 \times 8 + 3 \times 12}{2}$$

$$= \frac{20 \times 10 + 16 + 36}{2} = \frac{252}{2} = 126 \text{ square inches}$$

Regular Hexagon:



$A = 2.598s^2 = 2.598R^2 = 3.464r^2$
$R = s$ = radius of circumscribed circle = $1.155r$
r = radius of inscribed circle = $0.866s = 0.866R$
$s = R = 1.155r$

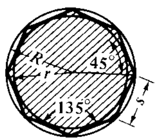
Example: The side s of a regular hexagon is 40 millimeters. Find the area and the radius r of the inscribed circle.

$$A = 2.598s^2 = 2.598 \times 40^2 = 2.598 \times 1600 = 4156.8 \text{ square millimeters}$$

$$r = 0.866s = 0.866 \times 40 = 34.64 \text{ millimeters}$$

Example: What is the length of the side of a hexagon that is drawn around a circle of 50 millimeters radius? — Here $r = 50$. Hence, $s = 1.155r = 1.155 \times 50 = 57.75$ millimeters

Regular Octagon:



A = area = $4.828s^2 = 2.828R^2 = 3.3\,14r^2$
R = radius of circumscribed circle = $1.307s = 1.082r$
r = radius of inscribed circle = $1.207s = 0.924R$
$s = 0.765R = 0.828r$

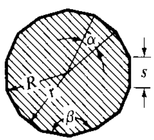
Example: Find the area and the length of the side of an octagon that is inscribed in a circle of 12 inches diameter.

Diameter of circumscribed circle = 12 inches; hence, $R = 6$ inches.

$$A = 2.828R^2 = 2.828 \times 6^2 = 2.828 \times 36 = 101.81 \text{ squre inches}$$

$$s = 0.765R = 0.765 \times 6 = 4.590 \text{ inches}$$

Regular Polygon:



A = area $\qquad n$ = number of sides

$\alpha = 360° \div n \qquad \beta = 180° - \alpha$

$$A = \frac{nsr}{2} = \frac{ns}{2}\sqrt{R^2 - \frac{s^2}{4}}$$

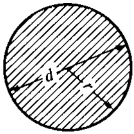
$$R = \sqrt{r^2 + \frac{s^2}{4}} \qquad r = \sqrt{R^2 - \frac{s^2}{4}} \qquad s = 2\sqrt{R^2 - r^2}$$

Example: Find the area of a polygon having 12 sides, inscribed in a circle of 8 centimeters radius. The length of the side s is 4.141 centimeters.

$$A = \frac{ns}{2}\sqrt{R^2 - \frac{s^2}{4}} = \frac{12 \times 4.141}{2}\sqrt{8^2 - \frac{4.141^2}{4}} = 24.846\sqrt{59.713}$$

$$= 24.846 \times 7.727 = 191.98 \text{ square centimeters}$$

Circle:



$$\text{Area} = A = \pi r^2 = 3.1416r^2 = 0.7854d^2$$

$$\text{Circumference} = C = 2\pi r = 6.832r = 3.1416d$$

$$r = C \div 6.2832 = \sqrt{A \div 3.1416} = 0.564\sqrt{A}$$

$$d = C \div 3.1416 = \sqrt{A \div 0.7854} = 1.128\sqrt{A}$$

Length of arc for center angle of 1° = 0.008727*d*

Length of arc for center angle of *n*° = 0.008727*nd*

Example: Find the area *A* and circumference *C* of a circle with a diameter of $2\frac{3}{4}$ inches.

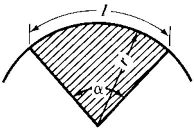
$$A = 0.7854d^2 = 0.7854 \times 2.75^2 = 0.7854 \times 2.75 \times 2.75 = 5.9396 \text{ square inches}$$

$$C = 3.1416d = 3.1416 \times 2.75 = 8.6394 \text{ inches}$$

Example: The area of a circle is 16.8 square inches. Find its diameter.

$$d = 1.128\sqrt{A} = 1.128\sqrt{16.8} = 1.128 \times 4.099 = 4.624 \text{ inches}$$

Circular Sector:



$$\text{Length of arc} = l = \frac{r \times \alpha \times 3.1416}{180} = 0.01745r\alpha = \frac{2A}{r}$$

$$\text{Area} = A = \tfrac{1}{2}rl = 0.008727\alpha r^2$$

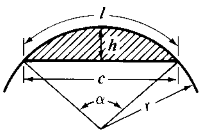
$$\text{Angle, in degrees} = \alpha = \frac{57.296\ l}{r} \qquad r = \frac{2A}{l} = \frac{57.296\ l}{\alpha}$$

Example: The radius of a circle is 35 millimeters, and angle α of a sector of the circle is 60 degrees. Find the area of the sector and the length of arc *l*.

$$A = 0.008727\alpha r^2 = 0.008727 \times 60 \times 35^2 = 641.41\text{mm}^2 = 6.41\text{cm}^2$$

$$l = 0.01745r\alpha = 0.01745 \times 35 \times 60 = 36.645 \text{ millimeters}$$

Circular Segment:



A = area $\qquad l$ = length of arc $\qquad \alpha$ = angle, in degrees

$$c = 2\sqrt{h(2r-h)} \qquad A = \tfrac{1}{2}[rl - c(r-h)]$$

$$r = \frac{c^2 + 4h^2}{8h} \qquad l = 0.01745r\alpha$$

$$h = r - \tfrac{1}{2}\sqrt{4r^2 - c^2} = r[1 - \cos(\alpha/2)] \qquad \alpha = \frac{57.296\ l}{r}$$

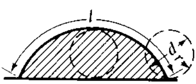
Example: The radius *r* is 60 inches and the height *h* is 8 inches. Find the length of the chord *c* .

$$c = 2\sqrt{h(2r-h)} = 2\sqrt{8 \times (2 \times 60 - 8)} = 2\sqrt{896} = 2 \times 29.93 = 59.86 \text{ inches}$$

Example: If *c* = 16, and *h* = 6 inches, what is the radius of the circle of which the segment is a part?

$$r = \frac{c^2 + 4h^2}{8h} = \frac{16^2 + 4 \times 6^2}{8 \times 6} = \frac{256 + 144}{48} = \frac{400}{48} = 8\tfrac{1}{3} \text{ inches}$$

Cycloid:



$$\text{Area} = A = 3\pi r^2 = 9.4248r^2 = 2.3562d^2$$

$$= 3 \times \text{area of generating circle}$$

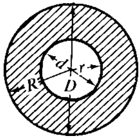
$$\text{Length of cycloid} = l = 8r = 4d$$

Example: The diameter of the generating circle of a cycloid is 6 inches. Find the length *l* of the cycloidal curve, and the area enclosed between the curve and the base line.

$$l = 4d = 4 \times 6 = 24 \text{ inches}$$

$$A = 2.3562d^2 = 2.3562 \times 6^2 = 84.82 \text{ square inches}$$

Circular Ring:



$$\text{Area} = A = \pi(R^2 - r^2) = 3.1416(R^2 - r^2)$$
$$= 3.1416(R + r)(R - r)$$
$$= 0.7845(D^2 - d^2) = 0.7854(D + d)(D - d)$$

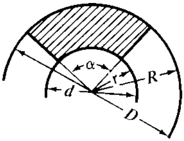
Example: Let the outside diameter $D = 12$ centimeters and the inside diameter $d = 8$ centimeters. Find the area of the ring.

$$A = 0.7854(D^2 - d^2) = 0.7854(12^2 - 8^2) = 0.7854(144 - 64) = 0.7854 \times 80$$
$$= 62.83 \text{ square centimeters}$$

By the alternative formula:

$$A = 0.7854(D + d)(D - d) = 0.7854(12 + 8)(12 - 8) = 0.7854 \times 20 \times 4$$
$$= 62.83 \text{ square centimeters}$$

Circular Ring Sector:



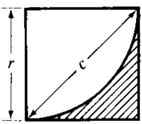
A = area α = angle, in degrees

$$A = \frac{\alpha\pi}{360}(R^2 - r^2) = 0.00873\alpha(R^2 - r^2)$$
$$= \frac{\alpha\pi}{4 \times 360}(D^2 - d^2) = 0.00218\alpha(D^2 - d^2)$$

Example: Find the area, if the outside radius $R = 5$ inches, the inside radius $r = 2$ inches, and $\alpha = 72$ degrees.

$$A = 0.00873\alpha(R^2 - r^2) = 0.00873 \times 72(5^2 - 2^2)$$
$$= 0.6286(25 - 4) = 0.6286 \times 21 = 13.2 \text{ square inches}$$

Spandrel or Fillet:



$$\text{Area} = A = r^2 - \frac{\pi r^2}{4} = 0.215r^2 = 0.1075c^2$$

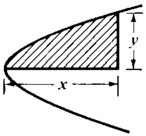
Example: Find the area of a spandrel, the radius of which is 0.7 inch.

$$A = 0.215r^2 = 0.215 \times 0.7^2 = 0.105 \text{ square inch}$$

Example: If chord c were given as 2.2 inches, what would be the area?

$$A = 0.1075c^2 = 0.1075 \times 2.2^2 = 0.520 \text{ square inch}$$

Parabola:



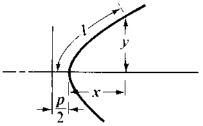
$$\text{Area} = A = \tfrac{2}{3}xy$$

(The area is equal to two-thirds of a rectangle which has x for its base and y for its height.)

Example: Let x in the illustration be 15 centimeters, and y, 9 centimeters. Find the area of the shaded portion of the parabola.

$$A = \tfrac{2}{3} \times xy = \tfrac{2}{3} \times 15 \times 9 = 10 \times 9 = 90 \text{ square centimeters}$$

Parabola:



$$l = \text{length of arc} = \frac{p}{2}\left[\sqrt{\frac{2x}{p}\left(1+\frac{2x}{p}\right)} + \ln\left(\sqrt{\frac{2x}{p}} + \sqrt{1+\frac{2x}{p}}\right)\right]$$

When x is small in proportion to y, the following is a close approximation:

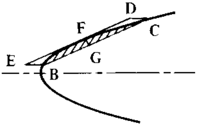
$$l = y\left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2 - \frac{2}{5}\left(\frac{x}{y}\right)^4\right] \text{ or } l = \sqrt{y^2 + \frac{4}{3}x^2}$$

Example: If $x = 2$ and $y = 24$ feet, what is the approximate length l of the parabolic curve?

$$l = y\left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2 - \frac{2}{5}\left(\frac{x}{y}\right)^4\right] = 24\left[1 + \frac{2}{3}\left(\frac{2}{24}\right)^2 - \frac{2}{5}\left(\frac{2}{24}\right)^4\right]$$

$$= 24\left[1 + \frac{2}{3} \times \frac{1}{144} - \frac{2}{5} \times \frac{1}{20{,}736}\right] = 24 \times 1.0046 = 24.11 \text{ feet}$$

Segment of Parabola:



$$\text{Area BFC} = A = \tfrac{2}{3} \text{ area of parallelogram BCDE}$$

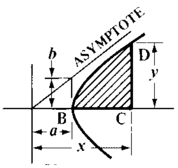
If FG is the height of the segment, measured at right angles to BC, then:

$$\text{Area of segment BFC} = \tfrac{2}{3}\text{BC} \times \text{FG}$$

Example: The length of the chord BC = 19.5 inches. The distance between lines BC and DE, measured at right angles to BC, is 2.25 inches. This is the height of the segment. Find the area.

$$\text{Area} = A = \tfrac{2}{3}\text{BC} \times \text{FG} = \tfrac{2}{3} \times 19.5 \times 2.25 = 29.25 \text{ square inches}$$

Hyperbola:



$$\text{Area BCD} = A = \frac{xy}{2} - \frac{ab}{2}\ln\left(\frac{x}{a} + \frac{y}{b}\right)$$

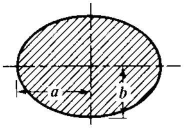
Example: The half-axes a and b are 3 and 2 inches, respectively. Find the area shown shaded in the illustration for $x = 8$ and $y = 5$.

Inserting the known values in the formula:

$$A = \frac{8 \times 5}{2} - \frac{3 \times 2}{2} \times \ln\left(\frac{8}{3} + \frac{5}{2}\right) = 20 - 3 \times \ln 5.167$$

$$= 20 - 3 \times 1.6423 = 20 - 4.927 = 15.073 \text{ square inches}$$

Ellipse:



$$\text{Area} = A = \pi ab = 3.1416ab$$

An approximate formula for the perimeter is

$$\text{Perimeter} = P = 3.1416\sqrt{2(a^2 + b^2)}$$

A closer approximation is $P = 3.1416\sqrt{2(a^2 + b^2) - \frac{(a-b)^2}{2.2}}$

Example: The larger or major axis is 200 millimeters. The smaller or minor axis is 150 millimeters. Find the area and the approximate circumference. Here, then, $a = 100$, and $b = 75$.

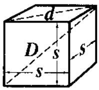
$$A = 3.1416ab = 3.1416 \times 100 \times 75 = 23{,}562 \text{ square millimeters} = 235.62 \text{ square centimeters}$$

$$P = 3.1416\sqrt{2(a^2 + b^2)} = 3.1416\sqrt{2(100^2 + 75^2)} = 3.1416\sqrt{2 \times 15{,}625}$$

$$= 3.1416\sqrt{31{,}250} = 3.1416 \times 176.78 = 555.37 \text{ millimeters} = (55.537 \text{ centimeters})$$

Volumes of Solids

Cube:



Diagonal of cube face $= d = s\sqrt{2}$

Diagonal of cube $= D = \sqrt{\frac{3d^2}{2}} = s\sqrt{3} = 1.732s$

Volume $= V = s^3$

$s = \sqrt[3]{V}$

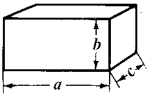
Example: The side of a cube equals 9.5 centimeters. Find its volume.

Volume $= V = s^3 = 9.5^3 = 9.5 \times 9.5 \times 9.5 = 857.375$ cubic centimeters

Example: The volume of a cube is 231 cubic centimeters. What is the length of the side?

$s = \sqrt[3]{V} = \sqrt[3]{231} = 6.136$ centimeters

Square Prism:



Volume $= V = abc$

$a = \frac{V}{bc}$ $\quad b = \frac{V}{ac}$ $\quad c = \frac{V}{ab}$

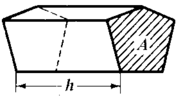
Example: In a square prism, $a = 6$, $b = 5$, $c = 4$. Find the volume.

$V = a \times b \times c = 6 \times 5 \times 4 = 120$ cubic inches

Example: How high should a box be made to contain 25 cubic feet, if it is 4 feet long and $2\frac{1}{2}$ feet wide? Here, $a = 4$, $c = 2.5$, and $V = 25$. Then,

$b = \text{depth} = \frac{V}{ac} = \frac{25}{4 \times 2.5} = \frac{25}{10} = 2.5$ feet

Prism:



V = volume

A = area of end surface

$V = h \times A$

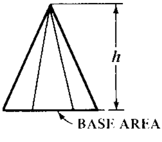
The area A of the end surface is found by the formulas for areas of plane figures on the preceding pages. Height h must be measured perpendicular to the end surface.

Example: A prism, having for its base a regular hexagon with a side s of 7.5 centimeters, is 25 centimeters high. Find the volume.

Area of hexagon $= A = 2.598s^2 = 2.598 \times 56.25 = 146.14$ square centimeters

Volume of prism $= h \times A = 25 \times 146.14 = 3653.5$ cubic centimeters

Pyramid:



Volume $= V = \frac{1}{3}h \times$ area of base

If the base is a regular polygon with n sides, and s = length of side, r = radius of inscribed circle, and R = radius of circumscribed circle, then:

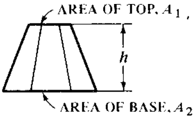
$$V = \frac{nsrh}{6} = \frac{nsh}{6}\sqrt{R^2 - \frac{s^2}{4}}$$

Example: A pyramid, having a height of 9 feet, has a base formed by a rectangle, the sides of which are 2 and 3 feet, respectively. Find the volume.

Area of base $= 2 \times 3 = 6$ square feet; $h = 9$ feet

Volume $= V = \frac{1}{3}h \times$ area of base $= \frac{1}{3} \times 9 \times 6 = 18$ cubic feet

Frustum of Pyramid:



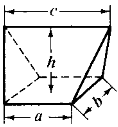
$$\text{Volume} = V = \frac{h}{3}(A_1 + A_2 + \sqrt{A_1 \times A_2})$$

Example: The pyramid in the previous example is cut off 4½ feet from the base, the upper part being removed. The sides of the rectangle forming the top surface of the frustum are, then, 1 and 1½ feet long, respectively. Find the volume of the frustum.

$$\text{Area of top} = A_1 = 1 \times 1\tfrac{1}{2} = 1\tfrac{1}{2} \text{ sq. ft.} \qquad \text{Area of base} = A_2 = 2 \times 3 = 6 \text{ sq. ft.}$$

$$V = \frac{4 \cdot 5}{3}(1.5 + 6 + \sqrt{1.5 \times 6}) = 1.5(7.5 + \sqrt{9}) = 1.5 \times 10.5 = 15.75 \text{ cubic feet}$$

Wedge:



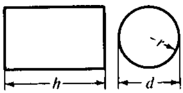
$$\text{Volume} = V = \frac{(2a + c)bh}{6}$$

Example: Let a = 4 inches, b = 3 inches, and c = 5 inches. The height h = 4.5 inches. Find the volume.

$$V = \frac{(2a + c)bh}{6} = \frac{(2 \times 4 + 5) \times 3 \times 4.5}{6} = \frac{(8 + 5) \times 13.5}{6}$$

$$= \frac{175.5}{6} = 29.25 \text{ cubic inches}$$

Cylinder:



$$\text{Volume} = V = 3.1416r^2h = 0.7854d^2h$$

$$\text{Area of cylindrical surface} = S = 6.2832rh = 3.1416dh$$

Total area A of cylindrical surface and end surfaces:

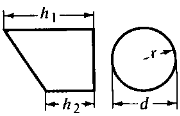
$$A = 6.2832r(r + h) = 3.1416d(\tfrac{1}{2}d + h)$$

Example: The diameter of a cylinder is 2.5 inches. The length or height is 20 inches. Find the volume and the area of the cylindrical surface S.

$$V = 0.7854d^2h = 0.7854 \times 2.5^2 \times 20 = 0.7854 \times 6.25 \times 20 = 98.17 \text{ cubic inches}$$

$$S = 3.1416dh = 3.1416 \times 2.5 \times 20 = 157.08 \text{ square inches}$$

Portion of Cylinder:



$$\text{Volume} = V = 1.5708r^2(h_1 + h_2)$$

$$= 0.3927d^2(h_1 + h_2)$$

$$\text{Cylindrical surface area} = S = 3.1416r(h_1 + h_2)$$

$$= 1.5708d(h_1 + h_2)$$

Example: A cylinder 125 millimeters in diameter is cut off at an angle, as shown in the illustration. Dimension h_1 = 150, and h_2 = 100 mm. Find the volume and the area S of the cylindrical surface.

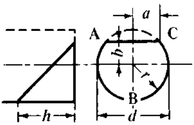
$$V = 0.3927d^2(h_1 + h_2) = 0.3927 \times 125^2 \times (150 + 100)$$

$$= 0.3927 \times 15{,}625 \times 250 = 1{,}533{,}984 \text{ cubic millimeters} = 1534 \text{ cm}^3$$

$$S = 1.5708d(h_1 + h_2) = 1.5708 \times 125 \times 250$$

$$= 49{,}087.5 \text{ square millimeters} = 490.9 \text{ square centimeters}$$

Portion of Cylinder:



$$\text{Volume} = V = \left(\tfrac{2}{3}a^3 \pm b \times \text{area ABC}\right)\frac{h}{r \pm b}$$

$$\text{Cylindrical surface area} = S = (ad \pm b \times \text{length of arc ABC})\frac{h}{r \pm b}$$

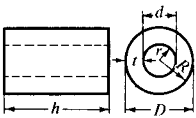
Use + when base area is larger, and – when base area is less than one-half the base circle.

Example: Find the volume of a cylinder so cut off that line *AC* passes through the center of the base circle — that is, the base area is a half-circle. The diameter of the cylinder = 5 inches, and the height h = 2 inches.

In this case, $a = 2.5$; $b = 0$; area $ABC = 0.5 \times 0.7854 \times 5^2 = 9.82$; $r = 2.5$.

$$V = \left(\frac{2}{3} \times 2.5^3 + 0 \times 9.82\right)\frac{2}{2.5 + 0} = \frac{2}{3} \times 15.625 \times 0.8 = 8.33 \text{ cubic inches}$$

Hollow Cylinder:

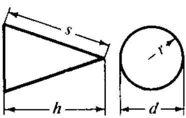


$$\begin{aligned}\text{Volume} = V &= 3.1416h(R^2 - r^2) = 0.7854h(D^2 - d^2)\\ &= 3.1416ht(2R - t) = 3.1416ht(D - t)\\ &= 3.1416ht(2r + t) = 3.1416ht(d + t)\\ &= 3.1416ht(R + r) = 1.5708ht(D + d)\end{aligned}$$

Example: A cylindrical shell, 28 centimeters high, is 36 centimeters in outside diameter, and 4 centimeters thick. Find its volume.

$$\begin{aligned}V &= 3.1416ht(D - t) = 3.1416 \times 28 \times 4(36 - 4) = 3.1416 \times 28 \times 4 \times 32\\ &= 11{,}259.5 \text{ cubic centimeters}\end{aligned}$$

Cone:



$$\text{Volume} = V = \frac{3.1416r^2h}{3} = 1.0472r^2h = 0.2618d^2h$$

$$\begin{aligned}\text{Conical surface area} = A &= 3.1416r\sqrt{r^2 + h^2} = 3.1416rs\\ &= 1.5708ds\end{aligned}$$

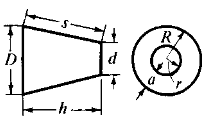
$$s = \sqrt{r^2 + h^2} = \sqrt{\frac{d^2}{4} + h^2}$$

Example: Find the volume and area of the conical surface of a cone, the base of which is a circle of 6 inches diameter, and the height of which is 4 inches.

$$V = 0.2618d^2h = 0.2618 \times 6^2 \times 4 = 0.2618 \times 36 \times 4 = 37.7 \text{ cubic inches}$$

$$\begin{aligned}A &= 3.1416r\sqrt{r^2 + h^2} = 3.1416 \times 3 \times \sqrt{3^2 + 4^2} = 9.4248 \times \sqrt{25}\\ &= 47.124 \text{ square inches}\end{aligned}$$

Frustum of Cone:



V = volume $\qquad A$ = area of conical surface

$$V = 1.0472h(R^2 + Rr + r^2) = 0.2618h(D^2 + Dd + d^2)$$

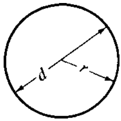
$$A = 3.1416s(R + r) = 1.5708s(D + d)$$

$$a = R - r \qquad s = \sqrt{a^2 + h^2} = \sqrt{(R - r)^2 + h^2}$$

Example: Find the volume of a frustum of a cone of the following dimensions: D = 8 centimeters; d = 4 centimeters; h = 5 centimeters.

$$\begin{aligned}V &= 0.2618 \times 5(8^2 + 8 \times 4 + 4^2) = 0.2618 \times 5(64 + 32 + 16)\\ &= 0.2618 \times 5 \times 112 = 146.61 \text{ cubic centimeters}\end{aligned}$$

Sphere:



$$\text{Volume} = V = \frac{4\pi r^3}{3} = \frac{\pi d^3}{6} = 4.1888r^3 = 0.5236d^3$$

$$\text{Surface area} = A = 4\pi r^2 = \pi d^2 = 12.5664r^2 = 3.1416d^2$$

$$r = \sqrt[3]{\frac{3V}{4\pi}} = 0.6024\sqrt[3]{V}$$

Example: Find the volume and the surface of a sphere 6.5 centimeters diameter.

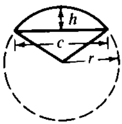
$$V = 0.5236d^3 = 0.5236 \times 6.5^3 = 0.5236 \times 6.5 \times 6.5 \times 6.5 = 143.79 \text{ cm}^3$$

$$A = 3.1416d^2 = 3.1416 \times 6.5^2 = 3.1416 \times 6.5 \times 6.5 = 132.73 \text{ cm}^2$$

Example: The volume of a sphere is 64 cubic centimeters. Find its radius.

$$r = 0.6204\sqrt[3]{64} = 0.6204 \times 4 = 2.4816 \text{ centimeters}$$

Spherical Sector:



$$V = \frac{2\pi r^2 h}{3} = 2.0944r^2h = \text{Volume}$$

$$A = 3.1416r(2h + \tfrac{1}{2}c)$$

$$= \text{total area of conical and spherical surface}$$

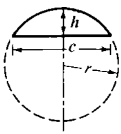
$$c = 2\sqrt{h(2r - h)}$$

Example: Find the volume of a sector of a sphere 6 inches in diameter, the height *h* of the sector being 1.5 inch. Also find the length of chord *c*. Here $r = 3$ and $h = 1.5$.

$$V = 2.0944r^2h = 2.0944 \times 3^2 \times 1.5 = 2.0944 \times 9 \times 1.5 = 28.27 \text{ cubic inches}$$

$$c = 2\sqrt{h(2r-h)} = 2\sqrt{1.5(2 \times 3 - 1.5)} = 2\sqrt{6.75} = 2 \times 2.598 = 5.196 \text{ inches}$$

Spherical Segment:



V = volume $\qquad$ A = area of spherical surface

$$V = 3.1416h^2\left(r - \frac{h}{3}\right) = 3.1416h\left(\frac{c^2}{8} + \frac{h^2}{6}\right)$$

$$A = 2\pi rh = 6.2832rh = 3.1416\left(\frac{c^2}{4} + h^2\right)$$

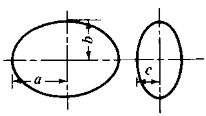
$$c = 2\sqrt{h(2r-h)}; \qquad r = \frac{c^2 + 4h^2}{8h}$$

Example: A segment of a sphere has the following dimensions: $h = 50$ millimeters; $c = 125$ millimeters. Find the volume *V* and the radius of the sphere of which the segment is a part.

$$V = 3.1416 \times 50 \times \left(\frac{125^2}{8} + \frac{50^2}{6}\right) = 157.08 \times \left(\frac{15{,}625}{8} + \frac{2500}{6}\right) = 372{,}247 \text{ mm}^3 = 372 \text{ cm}^3$$

$$r = \frac{125^2 + 4 \times 50^2}{8 \times 50} = \frac{15{,}625 + 10{,}000}{400} = \frac{25{,}625}{400} = 64 \text{ millimeters}$$

Ellipsoid:



$$\text{Volume} = V = \frac{4\pi}{3}abc = 4.1888abc$$

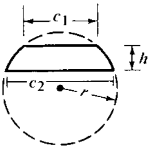
In an ellipsoid of revolution, or spheroid, where $c = b$:

$$V = 4.1888ab^2$$

Example: Find the volume of a spheroid in which $a = 5$, and $b = c = 1.5$ inches.

$$V = 4.1888 \times 5 \times 1.5^2 = 47.124 \text{ cubic inches}$$

Spherical Zone:



$$\text{Volume} = V = 0.5236h\left(\frac{3c_1^2}{4} + \frac{3c_2^2}{4} + h^2\right)$$

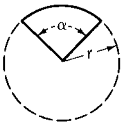
$$A = 2\pi rh = 6.2832rh = \text{area of spherical surface}$$

$$r = \sqrt{\frac{c_2^2}{4} + \left(\frac{c_2^2 - c_1^2 - 4h^2}{8h}\right)^2}$$

Example: In a spherical zone, let $c_1 = 3$; $c_2 = 4$; and $h = 1.5$ inch. Find the volume.

$$V = 0.5236 \times 1.5 \times \left(\frac{3 \times 3^2}{4} + \frac{3 \times 4^2}{4} + 1.5^2\right) = 0.5236 \times 1.5 \times \left(\frac{27}{4} + \frac{48}{4} + 2.25\right) = 16.493 \text{ in}^3$$

Spherical Wedge:



V = volume A = area of spherical surface

α = center angle in degrees

$$V = \frac{\alpha}{360} \times \frac{4\pi r^3}{3} = 0.0116\alpha r^3$$

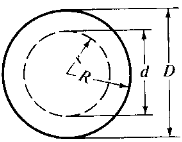
$$A = \frac{\alpha}{360} \times 4\pi r^2 = 0.0349\alpha r^2$$

Example: Find the area of the spherical surface and the volume of a wedge of a sphere. The diameter of the sphere is 100 millimeters, and the center angle α is 45 degrees.

$$V = 0.0116 \times 45 \times 50^3 = 0.0116 \times 45 \times 125{,}000 = 65{,}250 \text{ mm}^3 = 65.25 \text{ cm}^3$$

$$A = 0.0349 \times 45 \times 50^2 = 3926.25 \text{ square millimeters} = 39.26 \text{ cm}^2$$

Hollow Sphere:



V = volume of material used to make a hollow sphere

$$V = \frac{4\pi}{3}(R^3 - r^3) = 4.1888(R^3 - r^3)$$

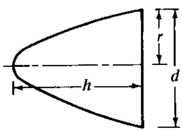
$$= \frac{\pi}{6}(D^3 - d^3) = 0.5236(D^3 - d^3)$$

Example: Find the volume of a hollow sphere, 8 inches in outside diameter, with a thickness of material of 1.5 inch.

Here $R = 4$; $r = 4 - 1.5 = 2.5$.

$$V = 4.1888(4^3 - 2.5^3) = 4.1888(64 - 15.625) = 4.1888 \times 48.375 = 202.63 \text{ cubic inches}$$

Paraboloid:



$$\text{Volume} = V = \tfrac{1}{2}\pi r^2 h = 0.3927d^2h$$

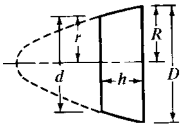
$$\text{Area} = A = \frac{2\pi}{3p}\left[\sqrt{\left(\frac{d^2}{4} + p^2\right)^3} - p^3\right]$$

$$\text{in which } p = \frac{d^2}{8h}$$

Example: Find the volume of a paraboloid in which $h = 300$ millimeters and $d = 125$ millimeters.

$$V = 0.3927d^2h = 0.3927 \times 125^2 \times 300 = 1{,}840{,}781 \text{ mm}^3 = 1{,}840.8 \text{ cm}^3$$

Paraboloidal Segment:



$$\text{Volume} = V = \frac{\pi}{2}h(R^2 + r^2) = 1.5708h(R^2 + r^2)$$

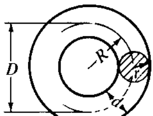
$$= \frac{\pi}{8}h(D^2 + d^2) = 0.3927h(D^2 + d^2)$$

Example: Find the volume of a segment of a paraboloid in which $D = 5$ inches, $d = 3$ inches, and $h = 6$ inches.

$$V = 0.3927h(D^2 + d^2) = 0.3927 \times 6 \times (5^2 + 3^2)$$

$$= 0.3927 \times 6 \times 34 = 80.11 \text{ cubic inches}$$

Torus:



$$\text{Volume} = V = 2\pi^2 Rr^2 = 19.739Rr^2$$

$$= \frac{\pi^2}{4}Dd^2 = 2.4674Dd^2$$

$$\text{Area of surface} = A = 4\pi^2 Rr = 39.478Rr$$

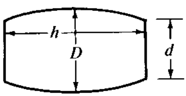
$$= \pi^2 Dd = 9.8696Dd$$

Example: Find the volume and area of surface of a torus in which $d = 1.5$ and $D = 5$ inches.

$$V = 2.4674 \times 5 \times 1.5^2 = 2.4674 \times 5 \times 2.25 = 27.76 \text{ cubic inches}$$

$$A = 9.8696 \times 5 \times 1.5 = 74.022 \text{ square inches}$$

Barrel:



V = approximate volume.

If the sides are bent to the arc of a circle:

$$V = \frac{1}{12}\pi h(2D^2 + d^2) = 0.262h(2D^2 + d^2)$$

If the sides are bent to the arc of a parabola:

$$V = 0.209h(2D^2 + Dd + \tfrac{3}{4}d^2)$$

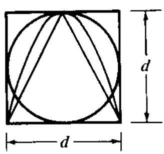
Example: Find the approximate contents of a barrel, the inside dimensions of which are $D = 60$ centimeters, $d = 50$ centimeters; $h = 120$ centimeters.

$$V = 0.262h(2D^2 + d^2) = 0.262 \times 120 \times (2 \times 60^2 + 50^2)$$

$$= 0.262 \times 120 \times (7200 + 2500) = 0.262 \times 120 \times 9700$$

$$= 304{,}968 \text{ cubic centimeters} = 0.305 \text{ cubic meter}$$

Ratio of Volumes:



If d = base diameter and height of a cone, a paraboloid and a cylinder, and the diameter of a sphere, then the volumes of these bodies are to each other as follows:

$$\text{Cone} : \text{paraboloid} : \text{sphere} : \text{cylinder} = \tfrac{1}{3} : \tfrac{1}{2} : \tfrac{2}{3} : 1$$

Example: Assume, as an example, that the diameter of the base of a cone, paraboloid, and cylinder is 2 inches, that the height is 2 inches, and that the diameter of a sphere is 2 inches. Then the volumes, written in formula form, are as follows:

Cone	Paraboloid	Sphere	Cylinder

$$\frac{3.1416 \times 2^2 \times 2}{12} : \frac{3.1416 \times (2p)^2 \times 2}{8} : \frac{3.1416 \times 2^3}{6} : \frac{3.1416 \times 2^2 \times 2}{4} = \tfrac{1}{3} : \tfrac{1}{2} : \tfrac{2}{3} : 1$$

Packing Circles in Circles and Rectangles

Diameter of Circle Enclosing a Given Number of Smaller Circles.—Four of many possible compact arrangements of circles within a circle are shown at A, B, C, and D in Fig. 1. To determine the diameter of the smallest enclosing circle for a particular number of enclosed circles all of the same size, three factors that influence the size of the enclosing circle should be considered. These are discussed in the paragraphs that follow, which are based on the article "How Many Wires Can Be Packed into a Circular Conduit," by Jacques Dutka, *Machinery,* October 1956.

1) *Arrangement of Center or Core Circles:* The four most common arrangements of center or core circles are shown cross-sectioned in Fig. 1. It may seem, offhand, that the "A" pattern would require the smallest enclosing circle for a given number of enclosed circles but this is not always the case since the most compact arrangement will, in part, depend on the number of circles to be enclosed.

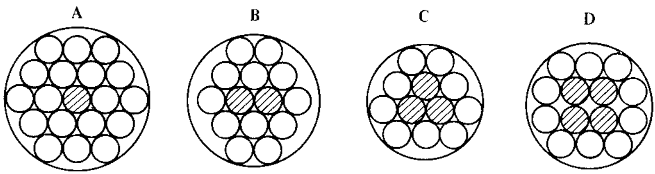


Fig. 1. Arrangements of Circles within a Circle

2) *Diameter of Enclosing Circle When Outer Layer of Circles Is Complete:* Successive, complete "layers" of circles may be placed around each of the central cores, Fig. 1, of 1, 2, 3, or 4 circles as the case may be. The number of circles contained in arrangements of complete "layers" around a central core of circles, as well as the diameter of the enclosing circle, may be obtained using the data in Table 1. Thus, for example, the "A" pattern in Fig. 1 shows, by actual count, a total of 19 circles arranged in two complete "layers" around a central core consisting of one circle; this agrees with the data shown in the left half of Table 1 for $n = 2$.

To determine the diameter of the enclosing circle, the data in the right half of Table 1 is used. Thus, for $n = 2$ and an "A" pattern, the diameter D is 5 times the diameter d of the enclosed circles.

3) *Diameter of Enclosing Circle When Outer Layer of Circles Is Not Complete:* In most cases, it is possible to reduce the size of the enclosing circle from that required if the outer layer were complete. Thus, for example, the "B" pattern in Fig. 1 shows that the central core consisting of 2 circles is surrounded by 1 complete layer of 8 circles and 1 partial, outer layer of 4 circles, so that the total number of circles enclosed is 14. If the outer layer were complete, then (from Table 1) the total number of enclosed circles would be 24 and the diameter of the enclosing circle would be $6d$; however, since the outer layer is composed of only 4 circles out of a possible 14 for a complete second layer, a smaller diameter of enclosing circle may be used. Table 2 shows that for a total of 14 enclosed circles arranged in a "B" pattern with the outer layer of circles incomplete, the diameter for the enclosing circle is $4.606d$.

Table 2 can be used to determine the smallest enclosing circle for a given number of circles to be enclosed by direct comparison of the "A," "B," and "C" columns. For data outside the range of Table 2, use the formulas in Dr. Dutka's article.

Table 1. Number of Circles Contained in Complete Layers of Circles and Diameter of Enclosing Circle (English or metric units)

No. Complete Layers Over Core, *n*	Number of Circles in Center Pattern							
	1	2	3	4	1	2	3	4
	Arrangement of Circles in Center Pattern (see Fig. 1)							
	"A"	"B"	"C"	"D"	"A"	"B"	"C"	"D"
	Number of Circles, *N*, Enclosed				Diameter, *D*, of Enclosing Circle[a]			
0	1	2	3	4	*d*	2*d*	2.155*d*	2.414*d*
1	7	10	12	14	3*d*	4*d*	4.055*d*	4.386*d*
2	19	24	27	30	5*d*	6*d*	6.033*d*	6.379*d*
3	37	44	48	52	7*d*	8*d*	8.024*d*	8.375*d*
4	61	70	75	80	9*d*	10*d*	10.018*d*	10.373*d*
5	91	102	108	114	11*d*	12*d*	12.015*d*	12.372*d*
n	[b]	[b]	[b]	[b]	[b]	[b]	[b]	[b]

[a] Diameter *D* is given in terms of *d*, the diameter of the enclosed circles.

[b] For *n* complete layers over core, the number of enclosed circles *N* for the "A" center pattern is $3n^2 + 3n + 1$; for "B," $3n^2 + 5n + 2$; for "C," $3n^2 + 6n + 3$; for "D," $3n^2 + 7n + 4$; while the diameter *D* of the enclosing circle for "A" center pattern is $(2n + 1)d$; for "B," $(2n + 2)d$; for "C," $(1 + 2\sqrt{n^2 + n + 1/3})d$ and for "D," $(1 + \sqrt{4n^2 + 5.644n + 2})d$.

Table 2. Factors for Determining Diameter, *D*, of Smallest Enclosing Circle for Various Numbers, *N*, of Enclosed Circles (English or metric units)

No. *N*	Center Circle Pattern			No. *N*	Center Circle Pattern			No. *N*	Center Circle Pattern		
	"A"	"B"	"C"		"A"	"B"	"C"		"A"	"B"	"C"
	Diameter Factor *K*				Diameter Factor *K*				Diameter Factor *K*		
2	3	2	...	34	7.001	7.083	7.111	66	9.718	9.545	9.327
3	3	2.733	2.155	35	7.001	7.245	7.111	67	9.718	9.545	9.327
4	3	2.733	3.310	36	7.001	7.245	7.111	68	9.718	9.545	9.327
5	3	3.646	3.310	37	7.001	7.245	7.430	69	9.718	9.661	9.327
6	3	3.646	3.310	38	7.929	7.245	7.430	70	9.718	9.661	10.019
7	3	3.646	4.056	39	7.929	7.558	7.430	71	9.718	9.889	10.019
8	4.465	3.646	4.056	40	7.929	7.558	7.430	72	9.718	9.889	10.019
9	4.465	4	4.056	41	7.929	7.558	7.430	73	9.718	9.889	10.019
10	4.465	4	4.056	42	7.929	7.558	7.430	74	10.166	9.889	10.019
11	4.465	4.606	4.056	43	7.929	8.001	8.024	75	10.166	10	10.019
12	4.465	4.606	4.056	44	8.212	8.001	8.024	76	10.166	10	10.238
13	4.465	4.606	5.164	45	8.212	8.001	8.024	77	10.166	10.540	10.238
14	5	4.606	5.164	46	8.212	8.001	8.024	78	10.166	10.540	10.238
15	5	5.359	5.164	47	8.212	8.001	8.024	79	10.166	10.540	10.452
16	5	5.359	5.164	48	8.212	8.001	8.024	80	10.166	10.540	10.452
17	5	5.359	5.164	49	8.212	8.550	8.572	81	10.166	10.540	10.452
18	5	5.359	5.164	50	8.212	8.550	8.572	82	10.166	10.540	10.452
19	5	5.583	5.619	51	8.212	8.550	8.572	83	10.166	10.540	10.452
20	6.292	5.583	5.619	52	8.212	8.550	8.572	84	10.166	10.540	10.452
21	6.292	5.583	5.619	53	8.212	8.811	8.572	85	10.166	10.644	10.866
22	6.292	5.583	6.034	54	8.212	8.811	8.572	86	11	10.644	10.866
23	6.292	6.001	6.034	55	8.212	8.811	9.083	87	11	10.644	10.866
24	6.292	6.001	6.034	56	9.001	8.811	9.083	88	11	10.644	10.866
25	6.292	6.197	6.034	57	9.001	8.938	9.083	89	11	10.849	10.866
26	6.292	6.197	6.034	58	9.001	8.938	9.083	90	11	10.849	10.866
27	6.292	6.568	6.034	59	9.001	8.938	9.083	91	11	10.849	11.067
28	6.292	6.568	6.774	60	9.001	8.938	9.083	92	11.393	10.849	11.067
29	6.292	6.568	6.774	61	9.001	9.186	9.083	93	11.393	11.149	11.067
30	6.292	6.568	6.774	62	9.718	9.186	9.083	94	11.393	11.149	11.067
31	6.292	7.083	7.111	63	9.718	9.186	9.083	95	11.393	11.149	11.067
32	7.001	7.083	7.111	64	9.718	9.186	9.327	96	11.393	11.149	11.067
33	7.001	7.083	7.111	65	9.718	9.545	9.327	97	11.393	11.441	11.264

Table 2. *(Continued)* **Factors for Determining Diameter, *D*, of Smallest Enclosing Circle for Various Numbers, *N*, of Enclosed Circles** (English or metric units)

No.	Center Circle Pattern			No.	Center Circle Pattern			No.	Center Circle Pattern		
	"A"	"B"	"C"		"A"	"B"	"C"		"A"	"B"	"C"
N	Diameter Factor *K*			*N*	Diameter Factor *K*			*N*	Diameter Factor *K*		
98	11.584	11.441	11.264	153	14.115	14	14.013	208	16.100	16	16.144
99	11.584	11.441	11.264	154	14.115	14	14.013	209	16.100	16.133	16.144
100	11.584	11.441	11.264	155	14.115	14.077	14.013	210	16.100	16.133	16.144
101	11.584	11.536	11.264	156	14.115	14.077	14.013	211	16.100	16.133	16.144
102	11.584	11.536	11.264	157	14.115	14.077	14.317	212	16.621	16.133	16.144
103	11.584	11.536	12.016	158	14.115	14.077	14.317	213	16.621	16.395	16.144
104	11.584	11.536	12.016	159	14.115	14.229	14.317	214	16.621	16.395	16.276
105	11.584	11.817	12.016	160	14.115	14.229	14.317	215	16.621	16.395	16.276
106	11.584	11.817	12.016	161	14.115	14.229	14.317	216	16.621	16.395	16.276
107	11.584	11.817	12.016	162	14.115	14.229	14.317	217	16.621	16.525	16.276
108	11.584	11.817	12.016	163	14.115	14.454	14.317	218	16.621	16.525	16.276
109	11.584	12	12.016	164	14.857	14.454	14.317	219	16.621	16.525	16.276
110	12.136	12	12.016	165	14.857	14.454	14.317	220	16.621	16.525	16.535
111	12.136	12.270	12.016	166	14.857	14.454	14.317	221	16.621	16.589	16.535
112	12.136	12.270	12.016	167	14.857	14.528	14.317	222	16.621	16.589	16.535
113	12.136	12.270	12.016	168	14.857	14.528	14.317	223	16.621	16.716	16.535
114	12.136	12.270	12.016	169	14.857	14.528	14.614	224	16.875	16.716	16.535
115	12.136	12.358	12.373	170	15	14.528	14.614	225	16.875	16.716	16.535
116	12.136	12.358	12.373	171	15	14.748	14.614	226	16.875	16.716	17.042
117	12.136	12.358	12.373	172	15	14.748	14.614	227	16.875	16.716	17.042
118	12.136	12.358	12.373	173	15	14.748	14.614	228	16.875	16.716	17.042
119	12.136	12.533	12.373	174	15	14.748	14.614	229	16.875	16.716	17.042
120	12.136	12.533	12.373	175	15	14.893	15.048	230	16.875	16.716	17.042
121	12.136	12.533	12.548	176	15	14.893	15.048	231	16.875	17.094	17.042
122	13	12.533	12.548	177	15	14.893	15.048	232	16.875	17.094	17.166
123	13	12.533	12.548	178	15	14.893	15.048	233	16.875	17.094	17.166
124	13	12.533	12.719	179	15	15.107	15.048	234	16.875	17.094	17.166
125	13	12.533	12.719	180	15	15.107	15.048	235	16.875	17.094	17.166
126	13	12.533	12.719	181	15	15.107	15.190	236	17	17.094	17.166
127	13	12.790	12.719	182	15	15.107	15.190	237	17	17.094	17.166
128	13.166	12.790	12.719	183	15	15.178	15.190	238	17	17.094	17.166
129	13.166	12.790	12.719	184	15	15.178	15.190	239	17	17.463	17.166
130	13.166	12.790	13.056	185	15	15.178	15.190	240	17	17.463	17.166
131	13.166	13.125	13.056	186	15	15.178	15.190	241	17	17.463	17.290
132	13.166	13.125	13.056	187	15	15.526	15.469	242	17.371	17.463	17.290
133	13.166	13.125	13.056	188	15.423	15.526	15.469	243	17.371	17.523	17.290
134	13.166	13.125	13.056	189	15.423	15.526	15.469	244	17.371	17.523	17.290
135	13.166	13.125	13.056	190	15.423	15.526	15.469	245	17.371	17.523	17.290
136	13.166	13.125	13.221	191	15.423	15.731	15.469	246	17.371	17.523	17.290
137	13.166	13.289	13.221	192	15.423	15.731	15.469	247	17.371	17.523	17.654
138	13.166	13.289	13.221	193	15.423	15.731	15.743	248	17.371	17.523	17.654
139	13.166	13.289	13.221	194	15.423	15.731	15.743	249	17.371	17.523	17.654
140	13.490	13.289	13.221	195	15.423	15.731	15.743	250	17.371	17.523	17.654
141	13.490	13.530	13.221	196	15.423	15.731	15.743	251	17.371	17.644	17.654
142	13.490	13.530	13.702	197	15.423	15.731	15.743	252	17.371	17.644	17.654
143	13.490	13.530	13.702	198	15.423	15.731	15.743	253	17.371	17.644	17.773
144	13.490	13.530	13.702	199	15.423	15.799	16.012	254	18.089	17.644	17.773
145	13.490	13.768	13.859	200	16.100	15.799	16.012	255	18.089	17.704	17.773
146	13.490	13.768	13.859	201	16.100	15.799	16.012	256	18.089	17.704	17.773
147	13.490	13.768	13.859	202	16.100	15.799	16.012	257	18.089	17.704	17.773
148	13.490	13.768	13.859	203	16.100	15.934	16.012	258	18.089	17.704	17.773
149	13.490	14	13.859	204	16.100	15.934	16.012	259	18.089	17.823	18.010
150	13.490	14	13.859	205	16.100	15.934	16.012	260	18.089	17.823	18.010
151	13.490	14	14.013	206	16.100	15.934	16.012	261	18.089	17.823	18.010
152	14.115	14	14.013	207	16.100	16	16.012	262	18.089	17.823	18.010

The diameter D of the enclosing circle is equal to the diameter factor, K, multiplied by d, the diameter of the enclosed circles, or $D = K \times d$. For example, if the number of circles to be enclosed, N, is 12, and the center circle arrangement is "C," then for $d = 1\frac{1}{2}$ inches, $D = 4.056 \times 1\frac{1}{2} = 6.084$ inches. If $d = 50$ millimeters, then $D = 4.056 \times 50 = 202.9$ millimeters.

Approximate Formula When Number of Enclosed Circles Is Large: When a large number of circles are to be enclosed, the arrangement of the center circles has little effect on the diameter of the enclosing circle. For numbers of circles greater than 10,000, the diameter of the enclosing circle may be calculated within 2 per cent from the formula $D = d(1 + \sqrt{N \div 0.907})$. In this formula, D = diameter of the enclosing circle; d = diameter of the enclosed circles; and N is the number of enclosed circles.

An alternative approach relates the area of each of the same-sized circles to be enclosed to the area of the enclosing circle (or container), as shown in Figs. 1 through 27. The table shows efficient ways for packing various numbers of circles N, from 2 up to 97.

In the table, D = the diameter of each circle to be enclosed, d = the diameter of the enclosing circle or container, and $\Phi = ND^2/d^2$ = ratio of the area of the N circles to the area of the enclosing circle or container, which is the packing efficiency. Cross-hatching in the diagrams indicates loose circles that may need packing constraints.

Data for Numbers of Circles in Circles

N	d/D	Φ	Fig.	N	d/D	Φ	Fig.
2	2.0000	0.500	1	17	4.7920	0.740	15
3	2.1547	0.646	2	18	4.8637	0.761	16
4	2.4142	0.686	3	19	4.8637	0.803	16
5	2.7013	0.685	4	20	5.1223	0.762	17
6	3.0000	0.667	5	21	5.2523	0.761	18
7	3.0000	0.778	5	22	5.4397	0.743	19
8	3.3048	0.733	6	23	5.5452	9.748	20
9	3.6131	0.689	7	24	5.6517	0.751	21
10	3.8130	0.688	8	25	5.7608	0.753	22
11	3.9238	0.714	9	31	6.2915	0.783	23
12	4.0296	0.739	10	37	6.7588	0.810	24
13	4.2361	0.724	11	55	8.2111	0.816	25
14	4.3284	0.747	12	61	8.6613	0.813	26
15	4.5214	0.734	13	97	11.1587	0.779	27
16	4.6154	0.751	14	...	...	...	...

Packing of large numbers of circles, such as the 97 in Fig. 27, may be approached by drawing a triangular pattern of circles, as shown in Fig. 28, which represents three circles near the center of the array. The point of a compass is then placed at A, B, or C, or anywhere within triangle ABC, and the radius of the compass is gradually enlarged until it encompasses the number of circles to be enclosed. As a first approximation of the diameter, $1.14D\sqrt{N}$ may be tried.

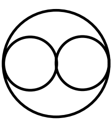


Fig. 1. $N = 2$

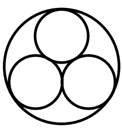


Fig. 2. $N = 3$

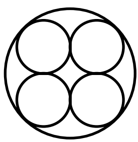


Fig. 3. $N = 4$

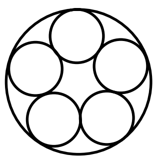


Fig. 4. $N = 5$

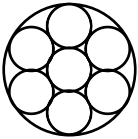


Fig. 5. $N = 7$

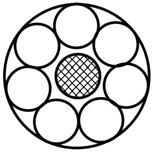


Fig. 6. $N = 8$

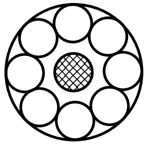


Fig. 7. $N = 9$

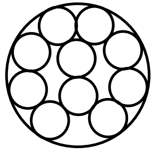


Fig. 8. $N = 10$

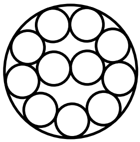


Fig. 9. $N = 11$

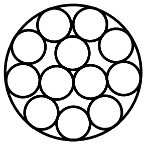


Fig. 10. $N = 12$

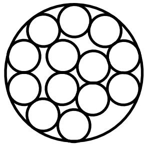


Fig. 11. $N = 13$

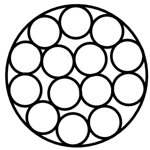


Fig. 12. $N = 14$

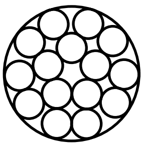


Fig. 13. $N = 15$

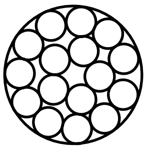


Fig. 14. $N = 16$

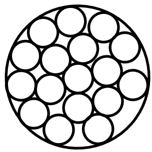


Fig. 15. $N = 17$

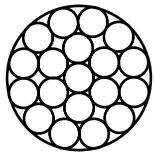


Fig. 16. $N = 19$

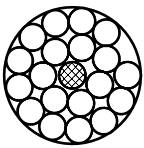


Fig. 17. $N = 20$

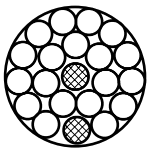


Fig. 18. $N = 21$

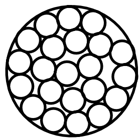


Fig. 19. $N = 22$

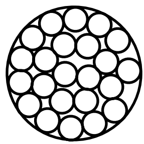


Fig. 20. $N = 23$

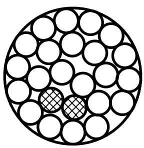


Fig. 21. $N = 24$

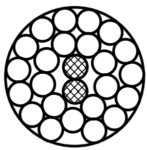


Fig. 22. $N = 25$

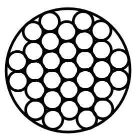


Fig. 23. $N = 31$

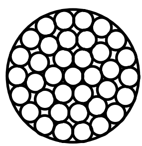


Fig. 24. $N = 37$

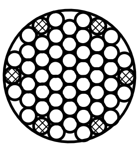


Fig. 25. $N = 55$

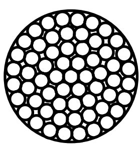


Fig. 26. $N = 61$

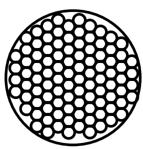


Fig. 27. $N = 97$

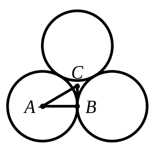


Fig. 28.

Circles within Rectangles.—For small numbers *N* of circles, packing (for instance, of cans) is less vital than for larger numbers and the number will usually govern the decision whether to use a rectangular or a triangular pattern, examples of which are seen in Figs. 29 and 30.

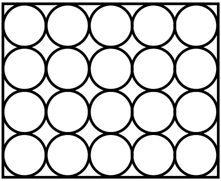


Fig. 29. Rectangular Pattern ($r = 4$, $c = 5$)

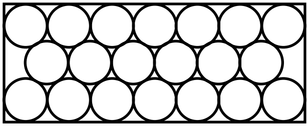


Fig. 30. Triangular Pattern ($r = 3$, $c = 7$)

If *D* is the can diameter and *H* its height, the arrangement in Fig. 29 will hold 20 circles or cans in a volume of $5D \times 4D \times H = 20D^2 H$. The arrangement in Fig. 30 will pack the same 20 cans into a volume of $7D \times 2.732D \times H = 19.124D^2 H$, a reduction of 4.4 per cent. When the ratio of H/D is less than 1.196 : 1, the rectangular pattern requires less surface area (therefore less material) for the six sides of the box, but for greater ratios, the triangular pattern is better. Some numbers, such as 19, can be accommodated only in a triangular pattern.

The following table shows possible patterns for 3 to 25 cans, where N = number of circles, P = pattern (*R* rectangular or *T* triangular), and *r* and *c* = numbers of rows and columns, respectively. The final table column shows the most economical application, where V = best volume, S = best surface area (sometimes followed by a condition on H/D). For the rectangular pattern, dimensions of the container are $rD \times cD$, and for the triangular pattern, the dimensions are) $\times [1 + (r + 70(-1))\sqrt{3}/2]$., or $cD^2[1 + 0.866(r-1)]$.

Numbers of Circles in Rectangular Arrangements

N	P	r	c	Application	N	P	r	c	Application
3	*T*	2	2	*V*, *S*	15	*R*	3	5	(*S*, $H/D > 0.038$)
						T	2	8	*V*, (*S*, $H/D < 0.038$)
4	*R*	2	2	*V*, *S*	16	*R*	4	4	*V*, *S*
5	*T*	3	2	*V*, *S*	17	*T*	3	6	*V*, *S*
6	*R*	2	3	*V*, *S*	18	*T*	5	4	*V*, *S*
7	*T*	2	4	*V*, *S*	19	*T*	2	10	*V*, *S*
8	*R*	4	2	*V*, (*S*, $H/D < 0.732$)	20	*R*	4	5	(*S*, $H/D > 1.196$)
	T	3	3	(*S*, $H/D > 0.732$)		*T*	3	7	*V*, (*S*, $H/D < 1.196$)
9	*R*	3	3	*V*, *S*	21	*R*	3	7	(*S*, $0.165 < H/D < 0.479$)
10	*R*	5	2	*V*, (*S*, $H/D > 1.976$)		*T*	6	4	(*S*, $H/D > 0.479$)
	T	4	3	(*S*, $H/D > 1.976$)		*T*	2	11	*V*, (*S*, $H/D < 0.165$)
11	*T*	3	4	*V*, *S*	22	*T*	4	6	*V*, *S*
12	*R*	3	4	*V*, *S*	23	*T*	5	5	(*S*, $H/D > 0.366$)
13	*T*	5	3	(*S*, $H/D > 0.236$)		*T*	3	8	*V*, (*S*, $H/D < 0.366$)
	T	2	7	*V*, (*S*, $H/D < 0.236$)	24	*R*	4	6	*V*, *S*
14	*T*	4	4	(*S*, $H/D > 5.464$)	25	*R*	5	5	(*S*, $H/D > 1.10$)
	T	3	5	*V*, (*S*, $H/D < 5.464$)		*T*	7	4	(S, $0.113 < H/D < 1.10$)
						T	2	13	*V*, (*S*, $H/D < 0.133$)

Formulas and Table for Regular Polygons.—The following formulas and table can be used to calculate the area, length of side, and radii of the inscribed and circumscribed circles of regular polygons (equal sided).

$$A = NS^2 \cot\alpha \div 4 = NR^2 \sin\alpha \cos\alpha = Nr^2 \tan\alpha$$

$$r = R\cos\alpha = (S\cot\alpha) \div 2 = \sqrt{(A \times \cot\alpha) \div N}$$

$$R = S \div (2\sin\alpha) = r \div \cos\alpha = \sqrt{A \div (N\sin\alpha\cos\alpha)}$$

$$S = 2R\sin\alpha = 2r\tan\alpha = 2\sqrt{(A \times \tan\alpha) \div N}$$

where N = number of sides; S = length of side; R = radius of circumscribed circle; r = radius of inscribed circle; A = area of polygon; and, $\alpha = 180° \div N$ = one-half center angle of one side (see *Regular Polygon* on page 62) .

Area, Length of Side, and Inscribed and Circumscribed Radii of Regular Polygons

No. of Sides	$\frac{A}{S^2}$	$\frac{A}{R^2}$	$\frac{A}{r^2}$	$\frac{R}{S}$	$\frac{R}{r}$	$\frac{S}{R}$	$\frac{S}{r}$	$\frac{r}{R}$	$\frac{r}{S}$
3	0.4330	1.2990	5.1962	0.5774	2.0000	1.7321	3.4641	0.5000	0.2887
4	1.0000	2.0000	4.0000	0.7071	1.4142	1.4142	2.0000	0.7071	0.5000
5	1.7205	2.3776	3.6327	0.8507	1.2361	1.1756	1.4531	0.8090	0.6882
6	2.5981	2.5981	3.4641	1.0000	1.1547	1.0000	1.1547	0.8660	0.8660
7	3.6339	2.7364	3.3710	1.1524	1.1099	0.8678	0.9631	0.9010	1.0383
8	4.8284	2.8284	3.3137	1.3066	1.0824	0.7654	0.8284	0.9239	1.2071
9	6.1818	2.8925	3.2757	1.4619	1.0642	0.6840	0.7279	0.9397	1.3737
10	7.6942	2.9389	3.2492	1.6180	1.0515	0.6180	0.6498	0.9511	1.5388
12	11.196	3.0000	3.2154	1.9319	1.0353	0.5176	0.5359	0.9659	1.8660
16	20.109	3.0615	3.1826	2.5629	1.0196	0.3902	0.3978	0.9808	2.5137
20	31.569	3.0902	3.1677	3.1962	1.0125	0.3129	0.3168	0.9877	3.1569
24	45.575	3.1058	3.1597	3.8306	1.0086	0.2611	0.2633	0.9914	3.7979
32	81.225	3.1214	3.1517	5.1011	1.0048	0.1960	0.1970	0.9952	5.0766
48	183.08	3.1326	3.1461	7.6449	1.0021	0.1308	0.1311	0.9979	7.6285
64	325.69	3.1365	3.1441	10.190	1.0012	0.0981	0.0983	0.9988	10.178

Example 1: A regular hexagon is inscribed in a circle of 6 inches diameter. Find the area and the radius of an inscribed circle. Here $R = 3$. From the table, area $A = 2.5981R^2 = 2.5981 \times 9 = 23.3829$ square inches. Radius of inscribed circle, $r = 0.866R = 0.866 \times 3 = 2.598$ inches.

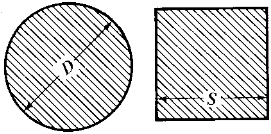
Example 2: An octagon is inscribed in a circle of 100 millimeters diameter. Thus $R = 50$. Find the area and radius of an inscribed circle. $A = 2.8284R^2 = 2.8284 \times 2500 = 7071$ mm^2 = 70.7 cm^2. Radius of inscribed circle, $r = 0.9239R = 09239 \times 50 = 46.195$ mm.

Example 3: Thirty-two bolts are to be equally spaced on the periphery of a bolt-circle, 16 inches in diameter. Find the chordal distance between the bolts. Chordal distance equals the side S of a polygon with 32 sides. $R = 8$. Hence, $S = 0.196R = 0.196 \times 8 = 1.568$ inch.

Example 4: Sixteen bolts are to be equally spaced on the periphery of a bolt-circle, 250 millimeters diameter. Find the chordal distance between the bolts. Chordal distance equals the side S of a polygon with 16 sides. $R = 125$. Thus, $S = 0.3902R = 0.3902 \times 125 = 48.775$ millimeters.

Tabulated Dimensions Of Geometric Shapes

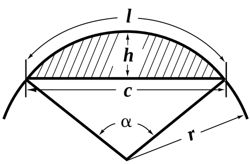
Diameters of Circles and Sides of Squares of Equal Area



The table below will be found useful for determining the diameter of a circle of an area equal to that of a square, the side of which is known, or for determining the side of a square which has an area equal to that of a circle, the area or diameter of which is known. For example, if the diameter of a circle is 17½ inches, it is found from the table that the side of a square of the same area is 15.51 inches.

Diam. of Circle, *D*	Side of Square, *S*	Area of Circle or Square	Diam. of Circle, *D*	Side of Square, *S*	Area of Circle or Square	Diam. of Circle, *D*	Side of Square, *S*	Area of Circle or Square
½	0.44	0.196	20½	18.17	330.06	40½	35.89	1288.25
1	0.89	0.785	21	18.61	346.36	41	36.34	1320.25
1½	1.33	1.767	21½	19.05	363.05	41½	36.78	1352.65
2	1.77	3.142	22	19.50	380.13	42	37.22	1385.44
2½	2.22	4.909	22½	19.94	397.61	42½	37.66	1418.63
3	2.66	7.069	23	20.38	415.48	43	38.11	1452.20
3½	3.10	9.621	23½	20.83	433.74	43½	38.55	1486.17
4	3.54	12.566	24	21.27	452.39	44	38.99	1520.53
4½	3.99	15.904	24½	21.71	471.44	44½	39.44	1555.28
5	4.43	19.635	25	22.16	490.87	45	39.88	1590.43
5½	4.87	23.758	25½	22.60	510.71	45½	40.32	1625.97
6	5.32	28.274	26	23.04	530.93	46	40.77	1661.90
6½	5.76	33.183	26½	23.49	551.55	46½	41.21	1698.23
7	6.20	38.485	27	23.93	572.56	47	41.65	1734.94
7½	6.65	44.179	27½	24.37	593.96	47½	42.10	1772.05
8	7.09	50.265	28	24.81	615.75	48	42.54	1809.56
8½	7.53	56.745	28½	25.26	637.94	48½	42.98	1847.45
9	7.98	63.617	29	25.70	660.52	49	43.43	1885.74
9½	8.42	70.882	29½	26.14	683.49	49½	43.87	1924.42
10	8.86	78.540	30	26.59	706.86	50	44.31	1963.50
10½	9.31	86.590	30½	27.03	730.62	50½	44.75	2002.96
11	9.75	95.033	31	27.47	754.77	51	45.20	2042.82
11½	10.19	103.87	31½	27.92	779.31	51½	45.64	2083.07
12	10.63	113.10	32	28.36	804.25	52	46.08	2123.72
12½	11.08	122.72	32½	28.80	829.58	52½	46.53	2164.75
13	11.52	132.73	33	29.25	855.30	53	46.97	2206.18
13½	11.96	143.14	33½	29.69	881.41	53½	47.41	2248.01
14	12.41	153.94	34	30.13	907.92	54	47.86	2290.22
14½	12.85	165.13	34½	30.57	934.82	54½	48.30	2332.83
15	13.29	176.71	35	31.02	962.11	55	48.74	2375.83
15½	13.74	188.69	35½	31.46	989.80	55½	49.19	2419.22
16	14.18	201.06	36	31.90	1017.88	56	49.63	2463.01
16½	14.62	213.82	36½	32.35	1046.35	56½	50.07	2507.19
17	15.07	226.98	37	32.79	1075.21	57	50.51	2551.76
17½	15.51	240.53	37½	33.23	1104.47	57½	50.96	2596.72
18	15.95	254.47	38	33.68	1134.11	58	51.40	2642.08
18½	16.40	268.80	38½	34.12	1164.16	58½	51.84	2687.83
19	16.84	283.53	39	34.56	1194.59	59	52.29	2733.97
19½	17.28	298.65	39½	35.01	1225.42	59½	52.73	2780.51
20	17.72	314.16	40	35.45	1256.64	60	53.17	2827.43

Segments of Circles for Radius = 1 (English or metric units)



Formulas for segments of circles are given on page 63. When the central angle α and radius *r* are known, the tables on these pages can be used to find the length of arc *l*, height of segment *h*, chord length *c*, and segment area *A*. When angle α and radius *r* are not known, but segment height *h* and chord length *c* are known or can be measured, the ratio *h*/*c* can be used to enter the table and find α, *l*, and *A* by linear interpolation. Radius *r* is found by the formula on page 63. The value of *l* is then multiplied by the radius *r* and the area *A* by r^2, the square of the radius.

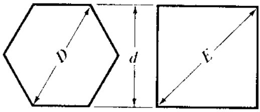
Angle α can be found thus with an accuracy of about 0.001 degree; arc length *l* with an error of about 0.02 per cent; and area *A* with an error ranging from about 0.02 per cent for the highest entry value of *h*/*c* to about 1 per cent for values of *h*/*c* of about 0.050. For lower values of *h*/*c*, and where greater accuracy is required, area *A* should be found by the formula on page 63.

θ, Deg.	*l*	*h*	*c*	*Area A*	*h*/*c*	θ, Deg.	*l*	*h*	*c*	*Area A*	*h*/*c*
1	0.01745	0.00004	0.01745	0.0000	0.00218	41	0.71558	0.06333	0.70041	0.0298	0.09041
2	0.03491	0.00015	0.03490	0.0000	0.00436	42	0.73304	0.06642	0.71674	0.0320	0.09267
3	0.05236	0.00034	0.05235	0.0000	0.00655	43	0.75049	0.06958	0.73300	0.0342	0.09493
4	0.06981	0.00061	0.06980	0.0000	0.00873	44	0.76794	0.07282	0.74921	0.0366	0.09719
5	0.08727	0.00095	0.08724	0.0001	0.01091	45	0.78540	0.07612	0.76537	0.0391	0.09946
6	0.10472	0.00137	0.10467	0.0001	0.01309	46	0.80285	0.07950	0.78146	0.0418	0.10173
7	0.12217	0.00187	0.12210	0.0002	0.01528	47	0.82030	0.08294	0.79750	0.0445	0.10400
8	0.13963	0.00244	0.13951	0.0002	0.01746	48	0.83776	0.08645	0.81347	0.0473	0.10628
9	0.15708	0.00308	0.15692	0.0003	0.01965	49	0.85521	0.09004	0.82939	0.0503	0.10856
10	0.17453	0.00381	0.17431	0.0004	0.02183	50	0.87266	0.09369	0.84524	0.0533	0.11085
11	0.19199	0.00460	0.19169	0.0006	0.02402	51	0.89012	0.09741	0.86102	0.0565	0.11314
12	0.20944	0.00548	0.20906	0.0008	0.02620	52	0.90757	0.10121	0.87674	0.0598	0.11543
13	0.22689	0.00643	0.22641	0.0010	0.02839	53	0.92502	0.10507	0.89240	0.0632	0.11773
14	0.24435	0.00745	0.24374	0.0012	0.03058	54	0.94248	0.10899	0.90798	0.0667	0.12004
15	0.26180	0.00856	0.26105	0.0015	0.03277	55	0.95993	0.11299	0.92350	0.0704	0.12235
16	0.27925	0.00973	0.27835	0.0018	0.03496	56	0.97738	0.11705	0.93894	0.0742	0.12466
17	0.29671	0.01098	0.29562	0.0022	0.03716	57	0.99484	0.12118	0.95432	0.0781	0.12698
18	0.31416	0.01231	0.31287	0.0026	0.03935	58	1.01229	0.12538	0.96962	0.0821	0.12931
19	0.33161	0.01371	0.33010	0.0030	0.04155	59	1.02974	0.12964	0.98485	0.0863	0.13164
20	0.34907	0.01519	0.34730	0.0035	0.04374	60	1.04720	0.13397	1.00000	0.0906	0.13397
21	0.36652	0.01675	0.36447	0.0041	0.04594	61	1.06465	0.13837	1.01508	0.0950	0.13632
22	0.38397	0.01837	0.38162	0.0047	0.04814	62	1.08210	0.14283	1.03008	0.0996	0.13866
23	0.40143	0.02008	0.39874	0.0053	0.05035	63	1.09956	0.14736	1.04500	0.1043	0.14101
24	0.41888	0.02185	0.41582	0.0061	0.05255	64	1.11701	0.15195	1.05984	0.1091	0.14337
25	0.43633	0.02370	0.43288	0.0069	0.05476	65	1.13446	0.15661	1.07460	0.1141	0.14574
26	0.45379	0.02563	0.44990	0.0077	0.05697	66	1.15192	0.16133	1.08928	0.1192	0.14811
27	0.47124	0.02763	0.46689	0.0086	0.05918	67	1.16937	0.16611	1.10387	0.1244	0.15048
28	0.48869	0.02970	0.48384	0.0096	0.06139	68	1.18682	0.17096	1.11839	0.1298	0.15287
29	0.50615	0.03185	0.50076	0.0107	0.06361	69	1.20428	0.17587	1.13281	0.1353	0.15525
30	0.52360	0.03407	0.51764	0.0118	0.06583	70	1.22173	0.18085	1.14715	0.1410	0.15765
31	0.54105	0.03637	0.53448	0.0130	0.06805	71	1.23918	0.18588	1.16141	0.1468	0.16005
32	0.55851	0.03874	0.55127	0.0143	0.07027	72	1.25664	0.19098	1.17557	0.1528	0.16246
33	0.57596	0.04118	0.56803	0.0157	0.07250	73	1.27409	0.19614	1.18965	0.1589	0.16488
34	0.59341	0.04370	0.58474	0.0171	0.07473	74	1.29154	0.20136	1.20363	0.1651	0.16730
35	0.61087	0.04628	0.60141	0.0186	0.07696	75	1.30900	0.20665	1.21752	0.1715	0.16973
36	0.62832	0.04894	0.61803	0.0203	0.07919	76	1.32645	0.21199	1.23132	0.1781	0.17216
37	0.64577	0.05168	0.63461	0.0220	0.08143	77	1.34390	0.21739	1.24503	0.1848	0.17461
38	0.66323	0.05448	0.65114	0.0238	0.08367	78	1.36136	0.22285	1.25864	0.1916	0.17706
39	0.68068	0.05736	0.66761	0.0257	0.08592	79	1.37881	0.22838	1.27216	0.1986	0.17952
40	0.69813	0.06031	0.68404	0.0277	0.08816	80	1.39626	0.23396	1.28558	0.2057	0.18199

Segments of Circles for Radius = 1 (English or metric units)

θ, Deg.	*l*	*h*	*c*	*Area A*	*h/c*	θ, Deg.	*l*	*h*	*c*	*Area A*	*h/c*
81	1.41372	0.23959	1.29890	0.2130	0.18446	131	2.28638	0.58531	1.81992	0.7658	0.32161
82	1.43117	0.24529	1.31212	0.2205	0.18694	132	2.30383	0.59326	1.82709	0.7803	0.32470
83	1.44862	0.25104	1.32524	0.2280	0.18943	133	2.32129	0.60125	1.83412	0.7950	0.32781
84	1.46608	0.25686	1.33826	0.2358	0.19193	134	2.33874	0.60927	1.84101	0.8097	0.33094
85	1.48353	0.26272	1.35118	0.2437	0.19444	135	2.35619	0.61732	1.84776	0.8245	0.33409
86	1.50098	0.26865	1.36400	0.2517	0.19696	136	2.37365	0.62539	1.85437	0.8395	0.33725
87	1.51844	0.27463	1.37671	0.2599	0.19948	137	2.39110	0.63350	1.86084	0.8546	0.34044
88	1.53589	0.28066	1.38932	0.2682	0.20201	138	2.40855	0.64163	1.86716	0.8697	0.34364
89	1.55334	0.28675	1.40182	0.2767	0.20456	139	2.42601	0.64979	1.87334	0.8850	0.34686
90	1.57080	0.29289	1.41421	0.2854	0.20711	140	2.44346	0.65798	1.87939	0.9003	0.35010
91	1.58825	0.29909	1.42650	0.2942	0.20967	141	2.46091	0.66619	1.88528	0.9158	0.35337
92	1.60570	0.30534	1.43868	0.3032	0.21224	142	2.47837	0.67443	1.89104	0.9314	0.35665
93	1.62316	0.31165	1.45075	0.3123	0.21482	143	2.49582	0.68270	1.89665	0.9470	0.35995
94	1.64061	0.31800	1.46271	0.3215	0.21741	144	2.51327	0.69098	1.90211	0.9627	0.36327
95	1.65806	0.32441	1.47455	0.3309	0.22001	145	2.53073	0.69929	1.90743	0.9786	0.36662
96	1.67552	0.33087	1.48629	0.3405	0.22261	146	2.54818	0.70763	1.91261	0.9945	0.36998
97	1.69297	0.33738	1.49791	0.3502	0.22523	147	2.56563	0.71598	1.91764	1.0105	0.37337
98	1.71042	0.34394	1.50942	0.3601	0.22786	148	2.58309	0.72436	1.92252	1.0266	0.37678
99	1.72788	0.35055	1.52081	0.3701	0.23050	149	2.60054	0.73276	1.92726	1.0428	0.38021
100	1.74533	0.35721	1.53209	0.3803	0.23315	150	2.61799	0.74118	1.93185	1.0590	0.38366
101	1.76278	0.36392	1.54325	0.3906	0.23582	151	2.63545	0.74962	1.93630	1.0753	0.38714
102	1.78024	0.37068	1.55429	0.4010	0.23849	152	2.65290	0.75808	1.94059	1.0917	0.39064
103	1.79769	0.37749	1.56522	0.4117	0.24117	153	2.67035	0.76655	1.94474	1.1082	0.39417
104	1.81514	0.38434	1.57602	0.4224	0.24387	154	2.68781	0.77505	1.94874	1.1247	0.39772
105	1.83260	0.39124	1.58671	0.4333	0.24657	155	2.70526	0.78356	1.95259	1.1413	0.40129
106	1.85005	0.39818	1.59727	0.4444	0.24929	156	2.72271	0.79209	1.95630	1.1580	0.40489
107	1.86750	0.40518	1.60771	0.4556	0.25202	157	2.74017	0.80063	1.95985	1.1747	0.40852
108	1.88496	0.41221	1.61803	0.4669	0.25476	158	2.75762	0.80919	1.96325	1.1915	0.41217
109	1.90241	0.41930	1.62823	0.4784	0.25752	159	2.77507	0.81776	1.96651	1.2084	0.41585
110	1.91986	0.42642	1.63830	0.4901	0.26028	160	2.79253	0.82635	1.96962	1.2253	0.41955
111	1.93732	0.43359	1.64825	0.5019	0.26306	161	2.80998	0.83495	1.97257	1.2422	0.42328
112	1.95477	0.44081	1.65808	0.5138	0.26585	162	2.82743	0.84357	1.97538	1.2592	0.42704
113	1.97222	0.44806	1.66777	0.5259	0.26866	163	2.84489	0.85219	1.97803	1.2763	0.43083
114	1.98968	0.45536	1.67734	0.5381	0.27148	164	2.86234	0.86083	1.98054	1.2934	0.43464
115	2.00713	0.46270	1.68678	0.5504	0.27431	165	2.87979	0.86947	1.98289	1.3105	0.43849
116	2.02458	0.47008	1.69610	0.5629	0.27715	166	2.89725	0.87813	1.98509	1.3277	0.44236
117	2.04204	0.47750	1.70528	0.5755	0.28001	167	2.91470	0.88680	1.98714	1.3449	0.44627
118	2.05949	0.48496	1.71433	0.5883	0.28289	168	2.93215	0.89547	1.98904	1.3621	0.45020
119	2.07694	0.49246	1.72326	0.6012	0.28577	169	2.94961	0.90415	1.99079	1.3794	0.45417
120	2.09440	0.50000	1.73205	0.6142	0.28868	170	2.96706	0.91284	1.99239	1.3967	0.45817
121	2.11185	0.50758	1.74071	0.6273	0.29159	171	2.98451	0.92154	1.99383	1.4140	0.46220
122	2.12930	0.51519	1.74924	0.6406	0.29452	172	3.00197	0.93024	1.99513	1.4314	0.46626
123	2.14675	0.52284	1.75763	0.6540	0.29747	173	3.01942	0.93895	1.99627	1.4488	0.47035
124	2.16421	0.53053	1.76590	0.6676	0.30043	174	3.03687	0.94766	1.99726	1.4662	0.47448
125	2.18166	0.53825	1.77402	0.6813	0.30341	175	3.05433	0.95638	1.99810	1.4836	0.47865
126	2.19911	0.54601	1.78201	0.6950	0.30640	176	3.07178	0.96510	1.99878	1.5010	0.48284
127	2.21657	0.55380	1.78987	0.7090	0.30941	177	3.08923	0.97382	1.99931	1.5184	0.48708
128	2.23402	0.56163	1.79759	0.7230	0.31243	178	3.10669	0.98255	1.99970	1.5359	0.49135
129	2.25147	0.56949	1.80517	0.7372	0.31548	179	3.12414	0.99127	1.99992	1.5533	0.49566
130	2.26893	0.57738	1.81262	0.7514	0.31854	180	3.14159	1.00000	2.00000	1.5708	0.50000

Distance Across Corners of Squares and Hexagons (English and metric units)



$$D = 1.154701d$$
$$E = 1.414214d$$

A desired value not given directly in the table can be obtained by the simple addition of two or more values taken directly from the table. Further values can be obtained by shifting the decimal point.

Example 1: Find *D* when *d* = 2 5/16 inches. From the table, 2 = 2.3094, and 5/16 = 0.3608. Therefore, D = 2.3094 + 0.3608 = 2.6702 inches.

Example 2: Find *E* when *d* = 20.25 millimeters. From the table, 20 = 28.2843; 0.2 = 0.2828; and 0.05 = 0.0707 (obtained by shifting the decimal point one place to the left at *d* = 0.5). Thus, *E* = 28.2843 + 0.2828 + 0.0707 = 28.6378 millimeters.

d	*D*	*E*	*d*	*D*	*E*	*d*	*D*	*E*	*d*	*D*	*E*
1/32	0.0361	0.0442	0.9	1.0392	1.2728	32	36.9504	45.2548	67	77.3650	94.7523
1/16	0.0722	0.0884	29/32	1.0464	1.2816	33	38.1051	46.6691	68	78.5197	96.1666
3/32	0.1083	0.1326	15/16	1.0825	1.3258	34	39.2598	48.0833	69	79.6744	97.5808
0.1	0.1155	0.1414	31/32	1.1186	1.3700	35	40.4145	49.4975	70	80.8291	98.9950
1/8	0.1443	0.1768	1.0	1.1547	1.4142	36	41.5692	50.9117	71	81.9838	100.409
5/32	0.1804	0.2210	2.0	2.3094	2.8284	37	42.7239	52.3259	72	83.1385	101.823
3/16	0.2165	0.2652	3.0	3.4641	4.2426	38	43.8786	53.7401	73	84.2932	103.238
0.2	0.2309	0.2828	4.0	4.6188	5.6569	39	45.0333	55.1543	74	85.4479	104.652
7/32	0.2526	0.3094	5.0	5.7735	7.0711	40	46.1880	56.5686	75	86.6026	106.066
1/4	0.2887	0.3536	6.0	6.9282	8.4853	41	47.3427	57.9828	76	87.7573	107.480
9/32	0.3248	0.3977	7.0	8.0829	9.8995	42	48.4974	59.3970	77	88.9120	108.894
0.3	0.3464	0.4243	8.0	9.2376	11.3137	43	49.6521	60.8112	78	90.0667	110.309
5/16	0.3608	0.4419	9.0	10.3923	12.7279	44	50.8068	62.2254	79	91.2214	111.723
11/32	0.3969	0.4861	10	11.5470	14.1421	45	51.9615	63.6396	80	92.3761	113.137
3/8	0.4330	0.5303	11	12.7017	15.5564	46	53.1162	65.0538	81	93.5308	114.551
0.4	0.4619	0.5657	12	13.8564	16.9706	47	54.2709	66.4681	82	94.6855	115.966
13/32	0.4691	0.5745	13	15.0111	18.3848	48	55.4256	67.8823	83	95.8402	117.380
7/16	0.5052	0.6187	14	16.1658	19.7990	49	56.5803	69.2965	84	96.9949	118.794
15/32	0.5413	0.6629	15	17.3205	21.2132	50	57.7351	70.7107	85	98.1496	120.208
0.5	0.5774	0.7071	16	18.4752	22.6274	51	58.8898	72.1249	86	99.3043	121.622
17/32	0.6134	0.7513	17	19.6299	24.0416	52	60.0445	73.5391	87	100.459	123.037
9/16	0.6495	0.7955	18	20.7846	25.4559	53	61.1992	74.9533	88	101.614	124.451
19/32	0.6856	0.8397	19	21.9393	26.8701	54	62.3539	76.3676	89	102.768	125.865
0.6	0.6928	0.8485	20	23.0940	28.2843	55	63.5086	77.7818	90	103.923	127.279
5/8	0.7217	0.8839	21	24.2487	29.6985	56	64.6633	79.1960	91	105.078	128.693
21/32	0.7578	0.9281	22	25.4034	31.1127	57	65.8180	80.6102	92	106.232	130.108
11/16	0.7939	0.9723	23	26.5581	32.5269	58	66.9727	82.0244	93	107.387	131.522
0.7	0.8083	0.9899	24	27.7128	33.9411	59	68.1274	83.4386	94	108.542	132.936
23/32	0.8299	1.0165	25	28.8675	35.3554	60	69.2821	84.8528	95	109.697	134.350
3/4	0.8660	1.0607	26	30.0222	36.7696	61	70.4368	86.2671	96	110.851	135.765
25/32	0.9021	1.1049	27	31.1769	38.1838	62	71.5915	87.6813	97	112.006	137.179
0.8	0.9238	1.1314	28	32.3316	39.5980	63	72.7462	89.0955	98	113.161	138.593
13/16	0.9382	1.1490	29	33.4863	41.0122	64	73.9009	90.5097	99	114.315	140.007
27/32	0.9743	1.1932	30	34.6410	42.4264	65	75.0556	91.9239	100	115.470	141.421
7/8	1.0104	1.2374	31	35.7957	43.8406	66	76.2103	93.3381	…	…	…

SOLUTION OF TRIANGLES

Any figure bounded by three straight lines is called a triangle. Any one of the three lines may be called the base, and the line drawn from the angle opposite the base at right angles to it is called the height or altitude of the triangle.

If all three sides of a triangle are of equal length, the triangle is called *equilateral*. Each of the three angles in an equilateral triangle equals 60 degrees. If two sides are of equal length, the triangle is an *isosceles* triangle. If one angle is a right or 90-degree angle, the triangle is a *right* or *right-angled* triangle. The side opposite the right angle is called the *hypotenuse*.

If all the angles are less than 90 degrees, the triangle is called an *acute* or *acute-angled* triangle. If one of the angles is larger than 90 degrees, the triangle is called an *obtuse-angled* triangle. Both acute and obtuse-angled triangles are known under the common name of *oblique-angled* triangles. The sum of the three angles in every triangle is 180 degrees.

The sides and angles of any triangle that are not known can be found when: 1) all the three sides; 2) two sides and one angle; and 3) one side and two angles are given.

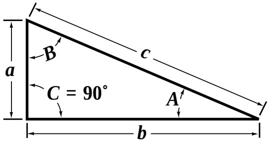
In other words, if a triangle is considered as consisting of six parts, three angles and three sides, the unknown parts can be determined when any three parts are given, provided at least one of the given parts is a side.

Functions of Angles

For every right triangle, a set of six ratios is defined; each is the length of one side of the triangle divided by the length of another side. The six ratios are the trigonometric (trig) functions sine, cosine, tangent, cosecant, secant, and cotangent (abbreviated sin, cos, tan, csc, sec, and cot). Trig functions are usually expressed in terms of an angle in degree or radian measure, as in cos 60° = 0.5. "Arc" in front of a trig function name, as in arcsin or arccos, means find the angle whose function value is given. For example, arcsin 0.5 = 30° means that 30° is the angle whose sin is equal to 0.5. Electronic calculators frequently use $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ to represent the arc functions.

Example: tan 53.1° = 1.332; arctan 1.332 = $\tan^{-1}$ 1.332 = 53.1° = 53° 6′

The *sine* of an angle equals the opposite side divided by the hypotenuse. Hence, $\sin B = b \div c$, and $\sin A = a \div c$.



The *cosine* of an angle equals the adjacent side divided by the hypotenuse. Hence, $\cos B = a \div c$, and $\cos A = b \div c$.

The *tangent* of an angle equals the opposite side divided by the adjacent side. Hence, $\tan B = b \div a$, and $\tan A = a \div b$.

The *cotangent* of an angle equals the adjacent side divided by the opposite side. Hence, $\cot B = a \div b$, and $\cot A = b \div a$.

The *secant* of an angle equals the hypotenuse divided by the adjacent side. Hence, $\sec B = c \div a$, and $\sec A = c \div b$.

The *cosecant* of an angle equals the hypotenuse divided by the opposite side. Hence, $\csc B = c \div b$, and $\csc A = c \div a$.

It should be noted that the functions of the angles can be found in this manner only when the triangle is right-angled.

If in a right-angled triangle (see preceding illustration), the lengths of the three sides are represented by *a*, *b*, and *c*, and the angles opposite each of these sides by *A*, *B*, and *C*, then the side *c* opposite the right angle is the hypotenuse; side *b* is called the *side adjacent* to angle *A* and is also the *side opposite* to angle *B*; side *a* is the side adjacent to angle *B* and the side opposite to angle A. The meanings of the various functions of angles can be explained with the aid of a right-angled triangle. Note that the cosecant, secant, and cotangent are the reciprocals of, respectively, the sine, cosine, and tangent.

The following relation exists between the angular functions of the two acute angles in a right-angled triangle: The sine of angle B equals the cosine of angle A; the tangent of angle B equals the cotangent of angle A, and *vice versa*. The sum of the two acute angles in a right-angled triangle always equals 90 degrees; hence, when one angle is known, the other can easily be found. When any two angles together make 90 degrees, one is called the *complement* of the other, and the sine of the one angle equals the cosine of the other, and the tangent of the one equals the cotangent of the other.

The Law of Sines.—In any triangle, any side is to the sine of the angle opposite that side as any other side is to the sine of the angle opposite that side. If a, b, and c are the sides, and A, B, and C their opposite angles, respectively, then:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}, \quad \text{so that:}$$

$$a = \frac{b \sin A}{\sin B} \quad \text{or} \quad a = \frac{c \sin A}{\sin C}$$

$$b = \frac{a \sin B}{\sin A} \quad \text{or} \quad b = \frac{c \sin B}{\sin C}$$

$$c = \frac{a \sin C}{\sin A} \quad \text{or} \quad c = \frac{b \sin C}{\sin B}$$

The Law of Cosines.—In any triangle, the square of any side is equal to the sum of the squares of the other two sides minus twice their product times the cosine of the included angle; or if a, b and c are the sides and A, B, and C are the opposite angles, respectively, then:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

These two laws, together with the proposition that the sum of the three angles equals 180 degrees, are the basis of all formulas relating to the solution of triangles.

Formulas for the solution of right-angled and oblique-angled triangles, arranged in tabular form, are given on the following pages.

Signs of Trigonometric Functions.—The diagram, *Signs of Trigonometric Functions, Fractions of π, and Degree–Radian Conversion* on page 92, shows the proper sign (+ or –) for the trigonometric functions of angles in each of the four quadrants, 0 to 90, 90 to 180, 180 to 270, and 270 to 360 degrees. Thus, the cosine of an angle between 90 and 180 degrees is negative; the sine of the same angle is positive.

Trigonometric Identities.—Trigonometric identities are formulas that show the relationship between different trigonometric functions. They may be used to change the form of some trigonometric expressions to simplify calculations. For example, if a formula has a term, $2\sin A\cos A$, the equivalent but simpler term $\sin 2A$ may be substituted. The identities that follow may themselves be combined or rearranged in various ways to form new identities.

Basic

$$\tan A = \frac{\sin A}{\cos A} = \frac{1}{\cot A} \qquad \sec A = \frac{1}{\cos A} \qquad \csc A = \frac{1}{\sin A}$$

Negative Angle

$\sin(-A) = -\sin A \quad \cos(-A) = \cos A \quad \tan(-A) = -\tan A$

Pythagorean

$\sin^2 A + \cos^2 A = 1 \quad 1 + \tan^2 A = \sec^2 A \quad 1 + \cot^2 A = \csc^2 A$

Sum and Difference of Angles

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \qquad \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\cot(A + B) = \frac{\cot A \cot B - 1}{\cot B + \cot A} \qquad \cot(A - B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B \qquad \sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B \qquad \cos(A - B) = \cos A \cos B - \sin A \sin B$$

Double-Angle

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\sin 2A = 2 \sin A \cos A \qquad \tan 2A = \frac{2 \tan A}{1 - \tan^2 A} = \frac{2}{\cot A - \tan A}$$

Half-Angle

$$\sin \tfrac{1}{2}A = \sqrt{\tfrac{1}{2}(1 - \cos A)} \qquad \cos \tfrac{1}{2}A = \sqrt{\tfrac{1}{2}(1 + \cos A)}$$

$$\tan \tfrac{1}{2}A = \sqrt{\frac{1 - \cos A}{1 + \cos A}} = \frac{1 - \cos A}{\sin A} = \frac{\sin A}{1 + \cos A}$$

Product-to-Sum

$$\sin A \cos B = \tfrac{1}{2}[\sin(A + B) + \sin(A - B)]$$

$$\cos A \cos B = \tfrac{1}{2}[\cos(A + B) + \cos(A - B)]$$

$$\sin A \sin B = \tfrac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$\tan A \tan B = \frac{\tan A + \tan B}{\cot A + \cot B}$$

Sum and Difference of Functions

$$\sin A + \sin B = 2[\sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B)]$$

$$\sin A - \sin B = 2[\sin \tfrac{1}{2}(A - B) \cos \tfrac{1}{2}(A + B)]$$

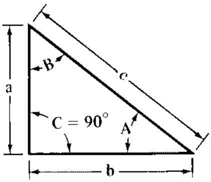
$$\cos A + \cos B = 2[\cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B)]$$

$$\cos A - \cos B = -2[\cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B)]$$

$$\tan A + \tan B = \frac{\sin(A + B)}{\cos A \cos B} \qquad \tan A - \tan B = \frac{\sin(A - B)}{\cos A \cos B}$$

$$\cot A + \cot B = \frac{\sin(B + A)}{\sin A \sin B} \qquad \cot A - \cot B = \frac{\sin(B - A)}{\sin A \sin B}$$

Solution of Right-Angled Triangles

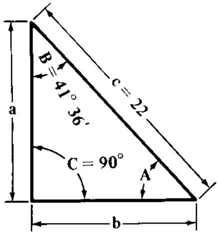


As shown in the illustration, the sides of the right-angled triangle are designated *a* and *b* and the hypotenuse, *c*. The angles opposite each of these sides are designated *A* and *B*, respectively.

Angle *C*, opposite the hypotenuse *c* is the right angle, and is therefore always one of the known quantities.

Sides and Angles Known	Formulas for Sides and Angles to be Found		
Side *a*; side *b*	$c = \sqrt{a^2 + b^2}$	$\tan A = \frac{a}{b}$	$B = 90° - A$
Side *a*; hypotenuse *c*	$b = \sqrt{c^2 - a^2}$	$\sin A = \frac{a}{c}$	$B = 90° - A$
Side *b*; hypotenuse *c*	$a = \sqrt{c^2 - b^2}$	$\sin B = \frac{b}{c}$	$A = 90° - B$
Hypotenuse *c*; angle *B*	$b = c \times \sin B$	$a = c \times \cos B$	$A = 90° - B$
Hypotenuse *c*; angle *A*	$b = c \times \cos A$	$a = c \times \sin A$	$B = 90° - A$
Side *b*; angle *B*	$c = \frac{b}{\sin B}$	$a = b \times \cot B$	$A = 90° - B$
Side *b*; angle *A*	$c = \frac{b}{\cos A}$	$a = b \times \tan A$	$B = 90° - A$
Side *a*; angle *B*	$c = \frac{a}{\cos B}$	$b = a \times \tan B$	$A = 90° - B$
Side *a*; angle *A*	$c = \frac{a}{\sin A}$	$b = a \times \cot A$	$B = 90° - A$

Examples of the Solution of Right-Angled Triangles (English and metric units)



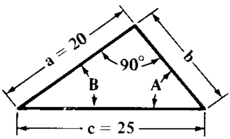
Hypotenuse and one angle known

c = 22 inches; B = 41° 36′.

$$a = c \times \cos B = 22 \times \cos 41^\circ 36' = 22 \times 0.74780 = 16.4516 \text{ inches}$$

$$b = c \times \sin B = 22 \times \sin 41^\circ 36' = 22 \times 0.66393 = 14.6065 \text{ inches}$$

$$A = 90^\circ - B = 90^\circ - 41^\circ 36' = 48^\circ 24'$$



Hypotenuse and one side known

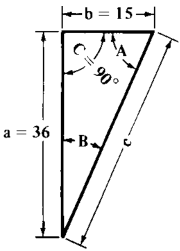
c = 25 centimeters; a = 20 centimeters.

$$b = \sqrt{c^2 - a^2} = \sqrt{25^2 - 20^2} = \sqrt{625 - 400} = \sqrt{225} = 15 \text{ centimeters}$$

$$\sin A = \frac{a}{c} = \frac{20}{25} = 0.8$$

Hence, $A = 53°8'$

$$B = 90° - A = 90° - 53°8' = 36°52'$$



Two sides known

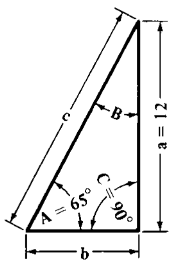
a = 36 inches; b = 15 inches.

$$c = \sqrt{a^2 + b^2} = \sqrt{36^2 + 15^2} = \sqrt{1296 + 225} = \sqrt{1521} = 39 \text{ inches}$$

$$\tan A = \frac{a}{b} = \frac{36}{15} = 2.4$$

Hence, $A = 67^\circ 23'$

$$B = 90^\circ - A = 90^\circ - 67^\circ 23' = 22^\circ 37'$$



One side and one angle known

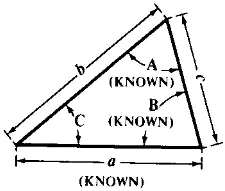
a = 12 meters; A = 65°.

$$c = \frac{a}{\sin A} = \frac{12}{\sin 65^\circ} = \frac{12}{0.90631} = 13.2405 \text{ meters}$$

$$b = a \times \cot A = 12 \times \cot 65^\circ = 12 \times 0.46631 = 5.5957 \text{ meters}$$

$$B = 90^\circ - A = 90^\circ - 65^\circ = 25^\circ$$

Solution of Oblique-Angled Triangles



One side and one angle known

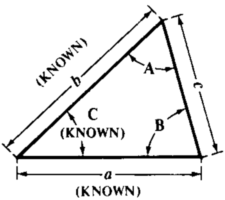
Call the known side *a*, the angle opposite it *A*, and the other known angle *B*. Then:

$C = 180° - (A + B)$; or if angles *B* and *C* are given, but not *A*, then $A = 180° - (B + C)$.

$$C = 180° - (A + B)$$

$$b = \frac{a \times \sin B}{\sin A} \qquad c = \frac{a \times \sin C}{\sin A}$$

$$\text{Area} = \frac{a \times b \times \sin C}{2}$$



Two sides and the angle between them known

Call the known sides *a* and *b*, and the known angle between them *C*. Then:

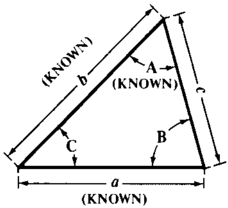
$$\tan A = \frac{a \times \sin C}{b - (a \times \cos C)}$$

$$B = 180° - (A + C) \qquad c = \frac{a \times \sin C}{\sin A}$$

Side *c* may also be found directly as below:

$$c = \sqrt{a^2 + b^2 - (2ab \times \cos C)}$$

$$\text{Area} = \frac{a \times b \times \sin C}{2}$$



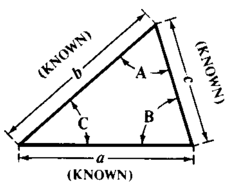
Two sides and the angle opposite one of the sides known

Call the known angle *A*, the side opposite it *a*, and the other known side *b*. Then:

$$\sin B = \frac{b \times \sin A}{a} \qquad C = 180° - (A + B)$$

$$c = \frac{a \times \sin C}{\sin A} \qquad \text{Area} = \frac{a \times b \times \sin C}{2}$$

If, in the above, angle $B >$ angle A but $<90°$, then a second solution B_2, C_2, c_2 exists for which: $B_2 = 180° - B$; $C_2 = 180° - (A + B_2)$; $c_2 = (a \times \sin C_2) \div \sin A$; area $= (a \times b \times \sin C_2) \div 2$. If $a \geq b$, then the first solution only exists. If $a < b \times \sin A$, then no solution exists.



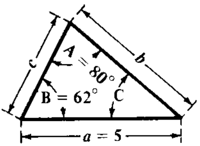
All three sides known

Call the sides *a*, *b*, and *c*, and the angles opposite them, *A*, *B*, and *C*. Then:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \qquad \sin B = \frac{b \times \sin A}{a}$$

$$C = 180° - (A + B) \qquad \text{Area} = \frac{a \times b \times \sin}{2}$$

Examples of the Solution of Oblique-Angled Triangles (English and metric units)



Side and angles known:

a = 5 centimeters; A = 80°; B = 62°

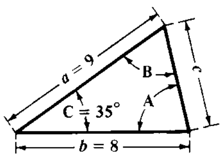
$$C = 180° - (80° + 62°) = 180° - 142° = 38°$$

$$b = \frac{a \times \sin B}{\sin A} = \frac{5 \times \sin 62°}{\sin 80°} = \frac{5 \times 0.88295}{0.98481} = 4.483$$

centimeters

$$c = \frac{a \times \sin C}{\sin A} = \frac{5 \times \sin 38°}{\sin 80°} = \frac{5 \times 0.61566}{0.98481} = 3.126$$

centimeters



Sides and angle known:

a = 9 inches; b = 8 inches; C = 35°.

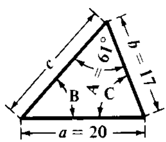
$$\tan A = \frac{a \times \sin C}{b - (a \times \cos C)} = \frac{9 \times \sin 35°}{8 - (9 \times \cos 35°)}$$

$$= \frac{9 \times 0.57358}{8 - (9 \times 0.81915)} = \frac{5.16222}{0.62765} = 8.2246$$

Hence, $A = 83°4'$

$$B = 180° - (A + C) = 180° - 118°4' = 61°$$

$$c = \frac{a \times \sin C}{\sin A} = \frac{9 \times 0.57358}{0.99269} = 5.2 \text{ inches}$$



Sides and angle known:

a = 20 centimeters; b = 17 centimeters; A = 61°.

$$\sin B = \frac{b \times \sin A}{a} = \frac{17 \times \sin 61°}{20}$$

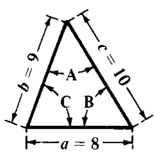
$$= \frac{17 \times 0.87462}{20} = 0.74343$$

Hence, $B = 48°1'$

$$C = 180° - (A + B) = 180° - 109°1' = 70°59'$$

$$c = \frac{a \times \sin C}{\sin A} = \frac{20 \times \sin 70°59'}{\sin 61°} = \frac{20 \times 0.94542}{0.87462}$$

$$= 21.62 \text{ centimeters}$$



Sides and angle known:

a = 8 inches; b = 9 inches; c = 10 inches.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{9^2 + 10^2 - 8^2}{2 \times 9 \times 10}$$

$$= \frac{81 + 100 - 64}{180} = \frac{117}{180} = 0.65000$$

Hence, $A = 49°27'$

$$\sin B = \frac{b \times \sin A}{a} = \frac{9 \times 0.75984}{8} = 0.85482$$

Hence, $B = 58°44'$

$$C = 180° - (A + B) = 180° - 108°11' = 71°49'$$

Conversion Tables of Angular Measure.—The accompanying tables of degrees, minutes, and seconds into radians; radians into degrees, minutes, and seconds; radians into degrees and decimals of a degree; and minutes and seconds into decimals of a degree and vice versa facilitate the conversion of measurements.

Example: The Degrees, Minutes, and Seconds into Radians table is used to find the number of radians in 324 degrees, 25 minutes, 13 seconds as follows:

300 degrees	=	5.235988 radians
20 degrees	=	0.349066 radian
4 degrees	=	0.069813 radian
25 minutes	=	0.007272 radian
13 seconds	=	0.000063 radian
324°25′13″	=	5.662202 radians

Example: The Radians into Degrees and Decimals of a Degree, and Radians into Degrees, Minutes and Seconds tables are used to find the number of decimal degrees or degrees, minutes and seconds in 0.734 radian as follows:

0.7 radian	=	40.1070 degrees	0.7 radian	=	40° 6′25″
0.03 radian	=	1.7189 degrees	0.03 radian	=	1°43′8″
0.004 radian	=	0.2292 degree	0.004 radian	=	0°13′45″
0.734 radian	=	42.0551 degrees	0.734 radian	=	41°62′78″ or 42°3′18″

Degrees, Minutes, and Seconds into Radians (Based on 180 degrees = π radians)

Degrees into Radians											
Deg.	Rad.	Deg.	Rad.	Deg.	Rad.	Deg.	Rad.	Deg.	Rad.	Deg.	Rad.
1000	17.453293	100	1.745329	10	0.174533	1	0.017453	0.1	0.001745	0.01	0.000175
2000	34.906585	200	3.490659	20	0.349066	2	0.034907	0.2	0.003491	0.02	0.000349
3000	52.359878	300	5.235988	30	0.523599	3	0.052360	0.3	0.005236	0.03	0.000524
4000	69.813170	400	6.981317	40	0.698132	4	0.069813	0.4	0.006981	0.04	0.000698
5000	87.266463	500	8.726646	50	0.872665	5	0.087266	0.5	0.008727	0.05	0.000873
6000	104.719755	600	10.471976	60	1.047198	6	0.104720	0.6	0.010472	0.06	0.001047
7000	122.173048	700	12.217305	70	1.221730	7	0.122173	0.7	0.012217	0.07	0.001222
8000	139.626340	800	13.962634	80	1.396263	8	0.139626	0.8	0.013963	0.08	0.001396
9000	157.079633	900	15.707963	90	1.570796	9	0.157080	0.9	0.015708	0.09	0.001571
10000	174.532925	1000	17.453293	100	1.745329	10	0.174533	1.0	0.017453	0.10	0.001745
Minutes into Radians											
Min.	Rad.	Min.	Rad.	Min.	Rad.	Min.	Rad.	Min.	Rad.	Min.	Rad.
1	0.000291	11	0.003200	21	0.006109	31	0.009018	41	0.011926	51	0.014835
2	0.000582	12	0.003491	22	0.006400	32	0.009308	42	0.012217	52	0.015126
3	0.000873	13	0.003782	23	0.006690	33	0.009599	43	0.012508	53	0.015417
4	0.001164	14	0.004072	24	0.006981	34	0.009890	44	0.012799	54	0.015708
5	0.001454	15	0.004363	25	0.007272	35	0.010181	45	0.013090	55	0.015999
6	0.001745	16	0.004654	26	0.007563	36	0.010472	46	0.013381	56	0.016290
7	0.002036	17	0.004945	27	0.007854	37	0.010763	47	0.013672	57	0.016581
8	0.002327	18	0.005236	28	0.008145	38	0.011054	48	0.013963	58	0.016872
9	0.002618	19	0.005527	29	0.008436	39	0.011345	49	0.014254	59	0.017162
10	0.002909	20	0.005818	30	0.008727	40	0.011636	50	0.014544	60	0.017453
Seconds into Radians											
Sec.	Rad.	Sec.	Rad.	Sec.	Rad.	Sec.	Rad.	Sec.	Rad.	Sec.	Rad.
1	0.000005	11	0.000053	21	0.000102	31	0.000150	41	0.000199	51	0.000247
2	0.000010	12	0.000058	22	0.000107	32	0.000155	42	0.000204	52	0.000252
3	0.000015	13	0.000063	23	0.000112	33	0.000160	43	0.000208	53	0.000257
4	0.000019	14	0.000068	24	0.000116	34	0.000165	44	0.000213	54	0.000262
5	0.000024	15	0.000073	25	0.000121	35	0.000170	45	0.000218	55	0.000267
6	0.000029	16	0.000078	26	0.000126	36	0.000175	46	0.000223	56	0.000271
7	0.000034	17	0.000082	27	0.000131	37	0.000179	47	0.000228	57	0.000276
8	0.000039	18	0.000087	28	0.000136	38	0.000184	48	0.000233	58	0.000281
9	0.000044	19	0.000092	29	0.000141	39	0.000189	49	0.000238	59	0.000286
10	0.000048	20	0.000097	30	0.000145	40	0.000194	50	0.000242	60	0.000291

Radians into Degrees and Decimals of a Degree
(Based on π radians = 180 degrees)

Rad.	Deg.	Rad.	Deg.	Rad.	Deg.	Rad.	Deg.	Rad.	Deg.	Rad.	Deg.
10	572.9578	1	57.2958	0.1	5.7296	0.01	0.5730	0.001	0.0573	0.0001	0.0057
20	1145.9156	2	114.5916	0.2	11.4592	0.02	1.1459	0.002	0.1146	0.0002	0.0115
30	1718.8734	3	171.8873	0.3	17.1887	0.03	1.7189	0.003	0.1719	0.0003	0.0172
40	2291.8312	4	229.1831	0.4	22.9183	0.04	2.2918	0.004	0.2292	0.0004	0.0229
50	2864.7890	5	286.4789	0.5	28.6479	0.05	2.8648	0.005	0.2865	0.0005	0.0286
60	3437.7468	6	343.7747	0.6	34.3775	0.06	3.4377	0.006	0.3438	0.0006	0.0344
70	4010.7046	7	401.0705	0.7	40.1070	0.07	4.0107	0.007	0.4011	0.0007	0.0401
80	4583.6624	8	458.3662	0.8	45.8366	0.08	4.5837	0.008	0.4584	0.0008	0.0458
90	5156.6202	9	515.6620	0.9	51.5662	0.09	5.1566	0.009	0.5157	0.0009	0.0516
100	5729.5780	10	572.9578	1.0	57.2958	0.10	5.7296	0.010	0.5730	0.0010	0.0573

Radians into Degrees, Minutes, and Seconds
(Based on π radians = 180 degrees)

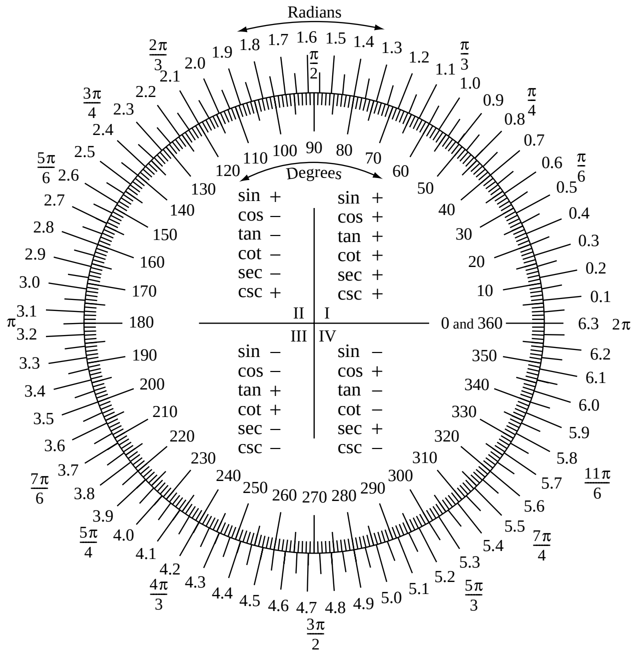
Rad.	Angle	Rad.	Angle	Rad.	Angle	Rad.	Angle	Rad.	Angle	Rad.	Angle
10	572°57′28″	1	57°17′45″	0.1	5°43′46″	0.01	0°34′23″	0.001	0°3′26″	0.0001	0°0′21″
20	1145°54′56″	2	114°35′30″	0.2	11°27′33″	0.02	1°8′45″	0.002	0°6′53″	0.0002	0°0′41″
30	1718°52′24″	3	171°53′14″	0.3	17°11′19″	0.03	1°43′8″	0.003	0°10′19″	0.0003	0°1′ 2″
40	2291°49′52″	4	229°10′59″	0.4	22°55′6″	0.04	2°17′31″	0.004	0°13′45″	0.0004	0°1′23″
50	2864°47′20″	5	286°28′44″	0.5	28°38′52″	0.05	2°51′53″	0.005	0°17′11″	0.0005	0°1′43″
60	3437°44′48″	6	343°46′29″	0.6	34°22′39″	0.06	3°26′16″	0.006	0°20′38″	0.0006	0°2′ 4″
70	4010°42′16″	7	401°4′14″	0.7	40°6′25″	0.07	4°0′39″	0.007	0°24′4″	0.0007	0°2′24″
80	4583°39′44″	8	458°21′58″	0.8	45°50′12″	0.08	4°35′1″	0.008	0°27′30″	0.0008	0°2′45″
90	5156°37′13″	9	515°39′43″	0.9	51°33′58″	0.09	5°9′24″	0.009	0°30′56″	0.0009	0°3′6″
100	5729°34′41″	10	572°57′28″	1.0	57°17′45″	0.10	5°43′46″	0.010	0°34′23″	0.0010	0°3′26″

Minutes and Seconds into Decimal of a Degree and Vice Versa
(Based on 1 second = 0.00027778 degree)

Minutes into Decimals of a Degree						Seconds into Decimals of a Degree					
Min.	Deg.	Min.	Deg.	Min.	Deg.	Sec.	Deg.	Sec.	Deg.	Sec.	Deg.
1	0.0167	21	0.3500	41	0.6833	1	0.0003	21	0.0058	41	0.0114
2	0.0333	22	0.3667	42	0.7000	2	0.0006	22	0.0061	42	0.0117
3	0.0500	23	0.3833	43	0.7167	3	0.0008	23	0.0064	43	0.0119
4	0.0667	24	0.4000	44	0.7333	4	0.0011	24	0.0067	44	0.0122
5	0.0833	25	0.4167	45	0.7500	5	0.0014	25	0.0069	45	0.0125
6	0.1000	26	0.4333	46	0.7667	6	0.0017	26	0.0072	46	0.0128
7	0.1167	27	0.4500	47	0.7833	7	0.0019	27	0.0075	47	0.0131
8	0.1333	28	0.4667	48	0.8000	8	0.0022	28	0.0078	48	0.0133
9	0.1500	29	0.4833	49	0.8167	9	0.0025	29	0.0081	49	0.0136
10	0.1667	30	0.5000	50	0.8333	10	0.0028	30	0.0083	50	0.0139
11	0.1833	31	0.5167	51	0.8500	11	0.0031	31	0.0086	51	0.0142
12	0.2000	32	0.5333	52	0.8667	12	0.0033	32	0.0089	52	0.0144
13	0.2167	33	0.5500	53	0.8833	13	0.0036	33	0.0092	53	0.0147
14	0.2333	34	0.5667	54	0.9000	14	0.0039	34	0.0094	54	0.0150
15	0.2500	35	0.5833	55	0.9167	15	0.0042	35	0.0097	55	0.0153
16	0.2667	36	0.6000	56	0.9333	16	0.0044	36	0.0100	56	0.0156
17	0.2833	37	0.6167	57	0.9500	17	0.0047	37	0.0103	57	0.0158
18	0.3000	38	0.6333	58	0.9667	18	0.0050	38	0.0106	58	0.0161
19	0.3167	39	0.6500	59	0.9833	19	0.0053	39	0.0108	59	0.0164
20	0.3333	40	0.6667	60	1.0000	20	0.0056	40	0.0111	60	0.0167

Example 1: Convert 11′37″ to decimals of a degree. From the left table, 11′ = 0.1833 degree. From the right table, 37″ = 0.0103 degree. Adding, 11′37″ = 0.1833 + 0.0103 = 0.1936 degree.

Example 2: Convert 0.1234 degree to minutes and seconds. From the left table, 0.1167 degree = 7′. Subtracting 0.1167 from 0.1234 gives 0.0067. From the right table, 0.0067 = 24″ so that 0.1234 = 7′24″.

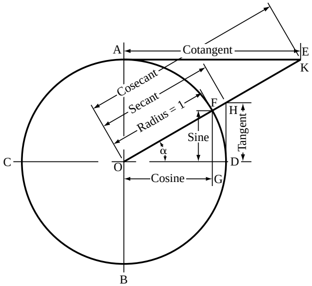


Signs of Trigonometric Functions, Fractions of π, and Degree–Radian Conversion

Graphic Illustrations of the Functions of Angles.—In graphically illustrating the functions of angles, it is assumed that all distances measured in the horizontal direction to the right of line *AB* are positive. Those measured horizontally to the left of *AB* are negative. All distances measured vertically, are positive above line *CD* and negative below it. It can then be readily seen that the sine is positive for all angles less than 180 degrees. For angles larger than 180 degrees, the sine would be measured below *CD*, and is negative. The cosine is positive up to 90 degrees, but for angles larger than 90 but less than 270 degrees, the cosine is measured to the left of line *AB* and is negative.

The table *Useful Relationships Among Angles* that follows is arranged to show directly whether the function of any given angle is positive or negative. It also gives the limits between which the numerical values of the function vary. For example, it will be seen from the table that the cosine of an angle between 90 and 180 degrees is negative, and that its value will be somewhere between 0 and – 1. In the same way, the cotangent of an angle between 180 and 270 degrees is positive and has a value between infinity and 0; in other words, the cotangent for 180 degrees is infinitely large and then the cotangent gradually decreases for increasing angles, so that the cotangent for 270 degrees equals 0.

The sine is positive for all angles up to 180 degrees. The cosine, tangent and cotangent for angles between 90 and 180 degrees, while they have the same numerical values as for angles from 0 to 90 degrees, are negative. These should be preceded by a minus sign; thus tan 123 degrees 20 minutes = –1.5204.



Tables of Trigonometric Functions.—The trigonometric (trig) tables on the following pages give numerical values for sine, cosine, tangent, and cotangent functions of angles from 0 to 90 degrees. Function values for all other angles can be obtained from the tables by applying the rules for signs of trigonometric functions and the useful relationships among angles given in the following. Secant and cosecant functions can be found from sec $A = 1/\cos A$ and csc $A = 1/\sin A$.

The trig tables are divided in half by a double line. The body of each half table consists of four labeled columns of data between columns listing angles. The angles listed to the left of the data increase, moving down the table, and angles listed to the right of the data increase, moving up the table. Labels above the data identify the trig functions corresponding to angles listed in the left column of each half table. Labels below the data correspond to angles listed in the right column of each half table. To find the value of a function for a particular angle, first locate the angle in the table, then find the appropriate function label across the top or bottom row of the table, and find the function value at the intersection of the angle row and label column. Angles opposite each other are complementary angles (i.e., their sum equals 90°) and are related. For example, sin 10° = cos 80° and cos 10° = sin 80°.

All the trig functions of angles between 0° and 90° have positive values. For other angles, consult the chart below to find the sign of the function in the quadrant where the angle is located. To determine trig functions of angles greater than 90° subtract 90, 180, 270, or 360 from the angle to get an angle less than 90° and use Table 1 to find the equivalent first-quadrant function and angle to look up in the trig tables.

Table 1. Useful Relationships Among Angles

Angle Function	θ	−θ	90° ± θ	180° ± θ	270° ± θ	360° ± θ
sin	sin θ	−sin θ	+cos θ	∓sin θ	−cos θ	±sin θ
cos	cos θ	+cos θ	∓sin θ	−cos θ	±sin θ	+cos θ
tan	tan θ	−tan θ	∓cot θ	±tan θ	∓cot θ	±tan θ
cot	cot θ	−cot θ	∓tan θ	±cot θ	∓tan θ	±cot θ
sec	sec θ	+sec θ	∓csc θ	−sec θ	±csc θ	+sec θ
csc	csc θ	−csc θ	+sec θ	∓csc θ	−sec θ	±csc θ

Examples: cos (270° − θ) = −sin θ; tan (90° + θ) = −cot θ.

Example: Find the cosine of 336°40′. The diagram in *Signs of Trigonometric Functions, Fractions of π, and Degree–Radian Conversion* shows that the cosine of every angle in Quadrant IV (270° to 360°) is positive. To find the angle and trig function to use when entering the trig table, subtract 270 from 336 to get cos 336°40′ = cos (270° + 66°40′) and then find the intersection of the cos row and the 270 ± θ column in Table 1. Because cos (270 ± θ) in the fourth quadrant is equal to ± sin θ in the first quadrant, find sin 66°40′ in the trig table. Therefore, cos 336°40′ = sin 66°40′ = 0.918216.

Trigonometric Functions of Angles from 0° to 15° and 75° to 90°

Angle	sin	cos	tan	cot		Angle	sin	cos	tan	cot	
0° 0′	0.000000	1.000000	0.000000	—	90° 0′	7° 30′	0.130526	0.991445	0.131652	7.595754	82° 30′
10	0.002909	0.999996	0.002909	343.7737	50	40	0.133410	0.991061	0.134613	7.428706	20
20	0.005818	0.999983	0.005818	171.8854	40	50	0.136292	0.990669	0.137576	7.268725	10
30	0.008727	0.999962	0.008727	114.5887	30	8° 0′	0.139173	0.990268	0.140541	7.115370	82° 0′
40	0.011635	0.999932	0.011636	85.93979	20	10	0.142053	0.989859	0.143508	6.968234	50
50	0.014544	0.999894	0.014545	68.75009	10	20	0.144932	0.989442	0.146478	6.826944	40
1° 0′	0.017452	0.999848	0.017455	57.28996	89° 0′	30	0.147809	0.989016	0.149451	6.691156	30
10	0.020361	0.999793	0.020365	49.10388	50	40	0.150686	0.988582	0.152426	6.560554	20
20	0.023269	0.999729	0.023275	42.96408	40	50	0.153561	0.988139	0.155404	6.434843	10
30	0.026177	0.999657	0.026186	38.18846	30	9° 0′	0.156434	0.987688	0.158384	6.313752	81° 0′
40	0.029085	0.999577	0.029097	34.36777	20	10	0.159307	0.987229	0.161368	6.197028	50
50	0.031992	0.999488	0.032009	31.24158	10	20	0.162178	0.986762	0.164354	6.084438	40
2° 0′	0.034899	0.999391	0.034921	28.63625	88° 0′	30	0.165048	0.986286	0.167343	5.975764	30
10	0.037806	0.999285	0.037834	26.43160	50	40	0.167916	0.985801	0.170334	5.870804	20
20	0.040713	0.999171	0.040747	24.54176	40	50	0.170783	0.985309	0.173329	5.769369	10
30	0.043619	0.999048	0.043661	22.90377	30	10° 0′	0.173648	0.984808	0.176327	5.671282	80° 0′
40	0.046525	0.998917	0.046576	21.47040	20	10	0.176512	0.984298	0.179328	5.576379	50
50	0.049431	0.998778	0.049491	20.20555	10	20	0.179375	0.983781	0.182332	5.484505	40
3° 0′	0.052336	0.998630	0.052408	19.08114	87° 0′	30	0.182236	0.983255	0.185339	5.395517	30
10	0.055241	0.998473	0.055325	18.07498	50	40	0.185095	0.982721	0.188349	5.309279	20
20	0.058145	0.998308	0.058243	17.16934	40	50	0.187953	0.982178	0.191363	5.225665	10
30	0.061049	0.998135	0.061163	16.34986	30	11° 0′	0.190809	0.981627	0.194380	5.144554	79° 0′
40	0.063952	0.997953	0.064083	15.60478	20	10	0.193664	0.981068	0.197401	5.065835	50
50	0.066854	0.997763	0.067004	14.92442	10	20	0.196517	0.980500	0.200425	4.989403	40
4° 0′	0.069756	0.997564	0.069927	14.30067	86° 0′	30	0.199368	0.979925	0.203452	4.915157	30
10	0.072658	0.997357	0.072851	13.72674	50	40	0.202218	0.979341	0.206483	4.843005	20
20	0.075559	0.997141	0.075775	13.19688	40	50	0.205065	0.978748	0.209518	4.772857	10
30	0.078459	0.996917	0.078702	12.70621	30	12° 0′	0.207912	0.978148	0.212557	4.704630	78° 0′
40	0.081359	0.996685	0.081629	12.25051	20	10	0.210756	0.977539	0.215599	4.638246	50
50	0.084258	0.996444	0.084558	11.82617	10	20	0.213599	0.976921	0.218645	4.573629	40
5° 0′	0.087156	0.996195	0.087489	11.43005	85° 0′	30	0.216440	0.976296	0.221695	4.510709	30
10	0.090053	0.995937	0.090421	11.05943	50	40	0.219279	0.975662	0.224748	4.449418	20
20	0.092950	0.995671	0.093354	10.71191	40	50	0.222116	0.975020	0.227806	4.389694	10
30	0.095846	0.995396	0.096289	10.38540	30	13° 0′	0.224951	0.974370	0.230868	4.331476	77° 0′
40	0.098741	0.995113	0.099226	10.07803	20	10	0.227784	0.973712	0.233934	4.274707	50
50	0.101635	0.994822	0.102164	9.788173	10	20	0.230616	0.973045	0.237004	4.219332	40
6° 0′	0.104528	0.994522	0.105104	9.514364	84° 0′	30	0.233445	0.972370	0.240079	4.165300	30
10	0.107421	0.994214	0.108046	9.255304	50	40	0.236273	0.971687	0.243157	4.112561	20
20	0.110313	0.993897	0.110990	9.009826	40	50	0.239098	0.970995	0.246241	4.061070	10
30	0.113203	0.993572	0.113936	8.776887	30	14° 0′	0.241922	0.970296	0.249328	4.010781	76° 0′
40	0.116093	0.993238	0.116883	8.555547	20	10	0.244743	0.969588	0.252420	3.961652	50
50	0.118982	0.992896	0.119833	8.344956	10	20	0.247563	0.968872	0.255516	3.913642	40
7° 0′	0.121869	0.992546	0.122785	8.144346	83° 0′	30	0.250380	0.968148	0.258618	3.866713	30
10	0.124756	0.992187	0.125738	7.953022	50	40	0.253195	0.967415	0.261723	3.820828	20
20	0.127642	0.991820	0.128694	7.770351	40	50	0.256008	0.966675	0.264834	3.775952	10
7° 30′	0.130526	0.991445	0.131652	7.595754	82° 30	15° 0′	0.258819	0.965926	0.267949	3.732051	75° 0′
	cos	sin	cot	tan		Angle	cos	sin	cot	tan	Angle

For angles 0° to 15° 0′ (angles found in a column to the left of the data), use the column labels at the top of the table; for angles 75° to 90° 0′ (angles found in a column to the right of the data), use the column labels at the bottom of the table.

Trigonometric Functions of Angles from 15° to 30° and 60° to 75°

Angle	sin	cos	tan	cot		Angle	sin	cos	tan	cot	
15° 0′	0.258819	0.965926	0.267949	3.732051	**75° 0′**	**22° 30′**	0.382683	0.923880	0.414214	2.414214	**67° 30**
10	0.261628	0.965169	0.271069	3.689093	**50**	**40**	0.385369	0.922762	0.417626	2.394489	**20**
20	0.264434	0.964404	0.274194	3.647047	**40**	**50**	0.388052	0.921638	0.421046	2.375037	**10**
30	0.267238	0.963630	0.277325	3.605884	**30**	**23° 0′**	0.390731	0.920505	0.424475	2.355852	**67° 0′**
40	0.270040	0.962849	0.280460	3.565575	**20**	**10**	0.393407	0.919364	0.427912	2.336929	**50**
50	0.272840	0.962059	0.283600	3.526094	**10**	**20**	0.396080	0.918216	0.431358	2.318261	**40**
16° 0′	0.275637	0.961262	0.286745	3.487414	**74° 0′**	**30**	0.398749	0.917060	0.434812	2.299843	**30**
10	0.278432	0.960456	0.289896	3.449512	**50**	**40**	0.401415	0.915896	0.438276	2.281669	**20**
20	0.281225	0.959642	0.293052	3.412363	**40**	**50**	0.404078	0.914725	0.441748	2.263736	**10**
30	0.284015	0.958820	0.296213	3.375943	**30**	**24° 0′**	0.406737	0.913545	0.445229	2.246037	**66° 0′**
40	0.286803	0.957990	0.299380	3.340233	**20**	**10**	0.409392	0.912358	0.448719	2.228568	**50**
50	0.289589	0.957151	0.302553	3.305209	**10**	**20**	0.412045	0.911164	0.452218	2.211323	**40**
17° 0′	0.292372	0.956305	0.305731	3.270853	**73° 0′**	**30**	0.414693	0.909961	0.455726	2.194300	**30**
10	0.295152	0.955450	0.308914	3.237144	**50**	**40**	0.417338	0.908751	0.459244	2.177492	**20**
20	0.297930	0.954588	0.312104	3.204064	**40**	**50**	0.419980	0.907533	0.462771	2.160896	**10**
30	0.300706	0.953717	0.315299	3.171595	**30**	**25° 0′**	0.422618	0.906308	0.466308	2.144507	**65° 0′**
40	0.303479	0.952838	0.318500	3.139719	**20**	**10**	0.425253	0.905075	0.469854	2.128321	**50**
50	0.306249	0.951951	0.321707	3.108421	**10**	**20**	0.427884	0.903834	0.473410	2.112335	**40**
18° 0′	0.309017	0.951057	0.324920	3.077684	**72° 0′**	**30**	0.430511	0.902585	0.476976	2.096544	**30**
10	0.311782	0.950154	0.328139	3.047492	**50**	**40**	0.433135	0.901329	0.480551	2.080944	**20**
20	0.314545	0.949243	0.331364	3.017830	**40**	**50**	0.435755	0.900065	0.484137	2.065532	**10**
30	0.317305	0.948324	0.334595	2.988685	**30**	**26° 0′**	0.438371	0.898794	0.487733	2.050304	**64° 0′**
40	0.320062	0.947397	0.337833	2.960042	**20**	**10**	0.440984	0.897515	0.491339	2.035256	**50**
50	0.322816	0.946462	0.341077	2.931888	**10**	**20**	0.443593	0.896229	0.494955	2.020386	**40**
19° 0′	0.325568	0.945519	0.344328	2.904211	**71° 0′**	**30**	0.446198	0.894934	0.498582	2.005690	**30**
10	0.328317	0.944568	0.347585	2.876997	**50**	**40**	0.448799	0.893633	0.502219	1.991164	**20**
20	0.331063	0.943609	0.350848	2.850235	**40**	**50**	0.451397	0.892323	0.505867	1.976805	**10**
30	0.333807	0.942641	0.354119	2.823913	**30**	**27° 0′**	0.453990	0.891007	0.509525	1.962611	**63° 0′**
40	0.336547	0.941666	0.357396	2.798020	**20**	**10**	0.456580	0.889682	0.513195	1.948577	**50**
50	0.339285	0.940684	0.360679	2.772545	**10**	**20**	0.459166	0.888350	0.516875	1.934702	**40**
20° 0′	0.342020	0.939693	0.363970	2.747477	**70° 0′**	**30**	0.461749	0.887011	0.520567	1.920982	**30**
10	0.344752	0.938694	0.367268	2.722808	**50**	**40**	0.464327	0.885664	0.524270	1.907415	**20**
20	0.347481	0.937687	0.370573	2.698525	**40**	**50**	0.466901	0.884309	0.527984	1.893997	**10**
30	0.350207	0.936672	0.373885	2.674621	**30**	**28° 0′**	0.469472	0.882948	0.531709	1.880726	**62° 0′**
40	0.352931	0.935650	0.377204	2.651087	**20**	**10**	0.472038	0.881578	0.535446	1.867600	**50**
50	0.355651	0.934619	0.380530	2.627912	**10**	**20**	0.474600	0.880201	0.539195	1.854616	**40**
21° 0′	0.358368	0.933580	0.383864	2.605089	**69° 0′**	**30**	0.477159	0.878817	0.542956	1.841771	**30**
10	0.361082	0.932534	0.387205	2.582609	**50**	**40**	0.479713	0.877425	0.546728	1.829063	**20**
20	0.363793	0.931480	0.390554	2.560465	**40**	**50**	0.482263	0.876026	0.550513	1.816489	**10**
30	0.366501	0.930418	0.393910	2.538648	**30**	**29° 0′**	0.484810	0.874620	0.554309	1.804048	**61° 0′**
40	0.369206	0.929348	0.397275	2.517151	**20**	**10**	0.487352	0.873206	0.558118	1.791736	**50**
50	0.371908	0.928270	0.400646	2.495966	**10**	**20**	0.489890	0.871784	0.561939	1.779552	**40**
22° 0′	0.374607	0.927184	0.404026	2.475087	**68° 0′**	**30**	0.492424	0.870356	0.565773	1.767494	**30**
10	0.377302	0.926090	0.407414	2.454506	**50**	**40**	0.494953	0.868920	0.569619	1.755559	**20**
20	0.379994	0.924989	0.410810	2.434217	**40**	**50**	0.497479	0.867476	0.573478	1.743745	**10**
22° 30	0.382683	0.923880	0.414214	2.414214	**67° 30**	**30° 0′**	0.500000	0.866025	0.577350	1.732051	**60° 0′**
	cos	sin	cot	tan	Angle		cos	sin	cot	tan	Angle

For angles 15° to 30° 0′ (angles found in a column to the left of the data), use the column labels at the top of the table; for angles 60° to 75° 0′ (angles found in a column to the right of the data), use the column labels at the bottom of the table.

Trigonometric Functions of Angles from 30° to 60°

Angle	sin	cos	tan	cot		Angle	sin	cos	tan	cot	
30° 0′	0.500000	0.866025	0.577350	1.732051	**60° 0′**	**37° 30′**	0.608761	0.793353	0.767327	1.303225	**52° 30′**
10	0.502517	0.864567	0.581235	1.720474	**50**	**40**	0.611067	0.791579	0.771959	1.295406	**20**
20	0.505030	0.863102	0.585134	1.709012	**40**	**50**	0.613367	0.789798	0.776612	1.287645	**10**
30	0.507538	0.861629	0.589045	1.697663	**30**	**38° 0′**	0.615661	0.788011	0.781286	1.279942	**52° 0′**
40	0.510043	0.860149	0.592970	1.686426	**20**	**10**	0.617951	0.786217	0.785981	1.272296	**50**
50	0.512543	0.858662	0.596908	1.675299	**10**	**20**	0.620235	0.784416	0.790697	1.264706	**40**
31° 0′	0.515038	0.857167	0.600861	1.664279	**59° 0′**	**30**	0.622515	0.782608	0.795436	1.257172	**30**
10	0.517529	0.855665	0.604827	1.653366	**50**	**40**	0.624789	0.780794	0.800196	1.249693	**20**
20	0.520016	0.854156	0.608807	1.642558	**40**	**50**	0.627057	0.778973	0.804979	1.242268	**10**
30	0.522499	0.852640	0.612801	1.631852	**30**	**39° 0′**	0.629320	0.777146	0.809784	1.234897	**51° 0′**
40	0.524977	0.851117	0.616809	1.621247	**20**	**10**	0.631578	0.775312	0.814612	1.227579	**50**
50	0.527450	0.849586	0.620832	1.610742	**10**	**20**	0.633831	0.773472	0.819463	1.220312	**40**
32° 0′	0.529919	0.848048	0.624869	1.600335	**58° 0′**	**30**	0.636078	0.771625	0.824336	1.213097	**30**
10	0.532384	0.846503	0.628921	1.590024	**50**	**40**	0.638320	0.769771	0.829234	1.205933	**20**
20	0.534844	0.844951	0.632988	1.579808	**40**	**50**	0.640557	0.767911	0.834155	1.198818	**10**
30	0.537300	0.843391	0.637070	1.569686	**30**	**40° 0′**	0.642788	0.766044	0.839100	1.191754	**50° 0′**
40	0.539751	0.841825	0.641167	1.559655	**20**	**10**	0.645013	0.764171	0.844069	1.184738	**50**
50	0.542197	0.840251	0.645280	1.549715	**10**	**20**	0.647233	0.762292	0.849062	1.177770	**40**
33° 0′	0.544639	0.838671	0.649408	1.539865	**57° 0′**	**30**	0.649448	0.760406	0.854081	1.170850	**30**
10	0.547076	0.837083	0.653551	1.530102	**50**	**40**	0.651657	0.758514	0.859124	1.163976	**20**
20	0.549509	0.835488	0.657710	1.520426	**40**	**50**	0.653861	0.756615	0.864193	1.157149	**10**
30	0.551937	0.833886	0.661886	1.510835	**30**	**41° 0′**	0.656059	0.754710	0.869287	1.150368	**49° 0′**
40	0.554360	0.832277	0.666077	1.501328	**20**	**10**	0.658252	0.752798	0.874407	1.143633	**50**
50	0.556779	0.830661	0.670284	1.491904	**10**	**20**	0.660439	0.750880	0.879553	1.136941	**40**
34° 0′	0.559193	0.829038	0.674509	1.482561	**56° 0′**	**30**	0.662620	0.748956	0.884725	1.130294	**30**
10	0.561602	0.827407	0.678749	1.473298	**50**	**40**	0.664796	0.747025	0.889924	1.123691	**20**
20	0.564007	0.825770	0.683007	1.464115	**40**	**50**	0.666966	0.745088	0.895151	1.117130	**10**
30	0.566406	0.824126	0.687281	1.455009	**30**	**42° 0′**	0.669131	0.743145	0.900404	1.110613	**48° 0′**
40	0.568801	0.822475	0.691572	1.445980	**20**	**10**	0.671289	0.741195	0.905685	1.104137	**50**
50	0.571191	0.820817	0.695881	1.437027	**10**	**20**	0.673443	0.739239	0.910994	1.097702	**40**
35° 0′	0.573576	0.819152	0.700208	1.428148	**55° 0′**	**30**	0.675590	0.737277	0.916331	1.091309	**30**
10	0.575957	0.817480	0.704551	1.419343	**50**	**40**	0.677732	0.735309	0.921697	1.084955	**20**
20	0.578332	0.815801	0.708913	1.410610	**40**	**50**	0.679868	0.733334	0.927091	1.078642	**10**
30	0.580703	0.814116	0.713293	1.401948	**30**	**43° 0′**	0.681998	0.731354	0.932515	1.072369	**47° 0′**
40	0.583069	0.812423	0.717691	1.393357	**20**	**10**	0.684123	0.729367	0.937968	1.066134	**50**
50	0.585429	0.810723	0.722108	1.384835	**10**	**20**	0.686242	0.727374	0.943451	1.059938	**40**
36° 0′	0.587785	0.809017	0.726543	1.376382	**54° 0′**	**30**	0.688355	0.725374	0.948965	1.053780	**30**
10	0.590136	0.807304	0.730996	1.367996	**50**	**40**	0.690462	0.723369	0.954508	1.047660	**20**
20	0.592482	0.805584	0.735469	1.359676	**40**	**50**	0.692563	0.721357	0.960083	1.041577	**10**
30	0.594823	0.803857	0.739961	1.351422	**30**	**44° 0′**	0.694658	0.719340	0.965689	1.035530	**46° 0′**
40	0.597159	0.802123	0.744472	1.343233	**20**	**10**	0.696748	0.717316	0.971326	1.029520	**50**
50	0.599489	0.800383	0.749003	1.335108	**10**	**20**	0.698832	0.715286	0.976996	1.023546	**40**
37° 0′	0.601815	0.798636	0.753554	1.327045	**53° 0′**	**30**	0.700909	0.713250	0.982697	1.017607	**30**
10	0.604136	0.796882	0.758125	1.319044	**50**	**40**	0.702981	0.711209	0.988432	1.011704	**20**
20	0.606451	0.795121	0.762716	1.311105	**40**	**50**	0.705047	0.709161	0.994199	1.005835	**10**
37° 30	0.608761	0.793353	0.767327	1.303225	**52° 30**	**45° 0′**	0.707107	0.707107	1.000000	1.000000	**45° 0′**
	cos	sin	cot	tan	Angle		cos	sin	cot	tan	Angle

For angles 30° to 45° 0′ (angles found in a column to the left of the data), use the column labels at the top of the table; for angles 45° to 60° 0′ (angles found in a column to the right of the data), use the column labels at the bottom of the table.

Using a Calculator to Find Trig Functions.—A scientific calculator is quicker and more accurate than tables for finding trig functions and angles corresponding to trig functions. On scientific calculators, the keys labeled **sin**, **cos**, and **tan** are used to find the common trig functions. The other functions can be found by using the same keys and the **1/*x*** key, noting that csc A = 1/sin A, sec A = 1/cos A, and cot A = 1/tan A. The specific keystrokes used will vary slightly from one calculator to another. To find the angle corresponding to a given trig function use the keys labeled $\mathbf{sin}^{-1}$, $\mathbf{cos}^{-1}$, and $\mathbf{tan}^{-1}$. On some other calculators, the **sin**, **cos**, and **tan** are used in combination with the **INV,** or inverse, key to find the number corresponding to a given trig function.

If a scientific calculator or computer is not available, tables are the easiest way to find trig values. However, trig function values can be calculated very accurately without a scientific calculator by using the following formulas:

$$\sin A = A - \frac{A^3}{3!} + \frac{A^5}{5!} - \frac{A^7}{7!} \pm \cdots \qquad \cos A = 1 - \frac{A^2}{2!} + \frac{A^4}{4!} - \frac{A^6}{6!} \pm \cdots$$

$$\sin^{-1} A = \frac{1}{2} \times \frac{A^3}{3} + \frac{1}{2} \times \frac{3}{4} \times \frac{A^5}{5} + \cdots \qquad \tan^{-1} A = A - \frac{A^3}{3} + \frac{A^5}{5} - \frac{A^7}{7} \pm \cdots$$

where the angle A is expressed in radians (convert degrees to radians by multiplying degrees by $\pi/180 = 0.0174533$). The three dots at the ends of the formulas indicate that the expression continues with more terms following the sequence established by the first few terms. Generally, calculating just three or four terms of the expression is sufficient for accuracy. In these formulas, a number followed by the symbol **!** is called a factorial (for example, 3! is three factorial). Except for 0!, which is defined as 1, a factorial is found by multiplying together all the integers greater than zero and less than or equal to the factorial number wanted. For example: $3! = 1 \times 2 \times 3 = 6$; $4! = 1 \times 2 \times 3 \times 4 = 24$; $7! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 = 5040$; etc.

Versed Sine and Versed Cosine.—These functions are sometimes used in formulas for segments of a circle and may be obtained using the relationships:

$$\text{versed } \sin\theta = 1 - \cos\theta;\ \text{versed } \cos\theta = 1 - \sin\theta.$$

Sevolute Functions.—Sevolute functions are used in calculating the form diameter of involute splines. They are computed by subtracting the involute function of an angle from the secant of the angle (1/cosine = secant). Thus, sevolute of 20 degrees = secant of 20 degrees – involute function of 20 degrees = 1.064178 – 0.014904 = 1.049274.

Involute Functions.—Involute functions are used in certain formulas relating to the design and measurement of gear teeth as well as measurement of threads over wires. See, for example, pages 1867 through 1870, 2080, and 2147.

The tables on the following pages provide values of involute functions for angles from 14 to 51 degrees in increments of 1 minute. These involute functions were calculated from the following formulas: Involute of $\theta = \tan\theta - \theta$, for θ in radians, and involute of $\theta = \tan\theta - \pi \times \theta/180$, for θ in degrees.

Example: For an angle of 14 degrees and 10 minutes, the involute function is found as follows: 10 minutes = 10⁄60 = 0.166666 degrees, 14 + 0.166666 = 14.166666 degree, so that the involute of 14.166666 degrees = tan 14.166666 – π × 14.166666⁄180 = 0.252420 – 0.247255 = 0.005165. This value is the same as that in the table *Involute Functions for Angles from 14 to 23 Degrees* for 14 degrees and 10 minutes. The same result would be obtained from using the conversion tables beginning on page 90 to convert 14 degrees and 10 minutes to radians and then applying the first of the formulas given above.

Involute Functions for Angles from 14 to 23 Degrees

	Degrees								
	14	15	16	17	18	19	20	21	22
Minutes	Involute Functions								
0	0.004982	0.006150	0.007493	0.009025	0.010760	0.012715	0.014904	0.017345	0.020054
1	0.005000	0.006171	0.007517	0.009052	0.010791	0.012750	0.014943	0.017388	0.020101
2	0.005018	0.006192	0.007541	0.009079	0.010822	0.012784	0.014982	0.017431	0.020149
3	0.005036	0.006213	0.007565	0.009107	0.010853	0.012819	0.015020	0.017474	0.020197
4	0.005055	0.006234	0.007589	0.009134	0.010884	0.012854	0.015059	0.017517	0.020244
5	0.005073	0.006255	0.007613	0.009161	0.010915	0.012888	0.015098	0.017560	0.020292
6	0.005091	0.006276	0.007637	0.009189	0.010946	0.012923	0.015137	0.017603	0.020340
7	0.005110	0.006297	0.007661	0.009216	0.010977	0.012958	0.015176	0.017647	0.020388
8	0.005128	0.006318	0.007686	0.009244	0.011008	0.012993	0.015215	0.017690	0.020436
9	0.005146	0.006340	0.007710	0.009272	0.011039	0.013028	0.015254	0.017734	0.020484
10	0.005165	0.006361	0.007735	0.009299	0.011071	0.013063	0.015293	0.017777	0.020533
11	0.005184	0.006382	0.007759	0.009327	0.011102	0.013098	0.015333	0.017821	0.020581
12	0.005202	0.006404	0.007784	0.009355	0.011133	0.013134	0.015372	0.017865	0.020629
13	0.005221	0.006425	0.007808	0.009383	0.011165	0.013169	0.015411	0.017908	0.020678
14	0.005239	0.006447	0.007833	0.009411	0.011196	0.013204	0.015451	0.017952	0.020726
15	0.005258	0.006469	0.007857	0.009439	0.011228	0.013240	0.015490	0.017996	0.020775
16	0.005277	0.006490	0.007882	0.009467	0.011260	0.013275	0.015530	0.018040	0.020824
17	0.005296	0.006512	0.007907	0.009495	0.011291	0.013311	0.015570	0.018084	0.020873
18	0.005315	0.006534	0.007932	0.009523	0.011323	0.013346	0.015609	0.018129	0.020921
19	0.005334	0.006555	0.007957	0.009552	0.011355	0.013382	0.015649	0.018173	0.020970
20	0.005353	0.006577	0.007982	0.009580	0.011387	0.013418	0.015689	0.018217	0.021019
21	0.005372	0.006599	0.008007	0.009608	0.011419	0.013454	0.015729	0.018262	0.021069
22	0.005391	0.006621	0.008032	0.009637	0.011451	0.013490	0.015769	0.018306	0.021118
23	0.005410	0.006643	0.008057	0.009665	0.011483	0.013526	0.015809	0.018351	0.021167
24	0.005429	0.006665	0.008082	0.009694	0.011515	0.013562	0.015850	0.018395	0.021217
25	0.005448	0.006687	0.008107	0.009722	0.011547	0.013598	0.015890	0.018440	0.021266
26	0.005467	0.006709	0.008133	0.009751	0.011580	0.013634	0.015930	0.018485	0.021316
27	0.005487	0.006732	0.008158	0.009780	0.011612	0.013670	0.015971	0.018530	0.021365
28	0.005506	0.006754	0.008183	0.009808	0.011644	0.013707	0.016011	0.018575	0.021415
29	0.005525	0.006776	0.008209	0.009837	0.011677	0.013743	0.016052	0.018620	0.021465
30	0.005545	0.006799	0.008234	0.009866	0.011709	0.013779	0.016092	0.018665	0.021514
31	0.005564	0.006821	0.008260	0.009895	0.011742	0.013816	0.016133	0.018710	0.021564
32	0.005584	0.006843	0.008285	0.009924	0.011775	0.013852	0.016174	0.018755	0.021614
33	0.005603	0.006866	0.008311	0.009953	0.011807	0.013889	0.016215	0.018800	0.021665
34	0.005623	0.006888	0.008337	0.009982	0.011840	0.013926	0.016255	0.018846	0.021715
35	0.005643	0.006911	0.008362	0.010011	0.011873	0.013963	0.016296	0.018891	0.021765
36	0.005662	0.006934	0.008388	0.010041	0.011906	0.013999	0.016337	0.018937	0.021815
37	0.005682	0.006956	0.008414	0.010070	0.011939	0.014036	0.016379	0.018983	0.021866
38	0.005702	0.006979	0.008440	0.010099	0.011972	0.014073	0.016420	0.019028	0.021916
39	0.005722	0.007002	0.008466	0.010129	0.012005	0.014110	0.016461	0.019074	0.021967
40	0.005742	0.007025	0.008492	0.010158	0.012038	0.014148	0.016502	0.019120	0.022018
41	0.005762	0.007048	0.008518	0.010188	0.012071	0.014185	0.016544	0.019166	0.022068
42	0.005782	0.007071	0.008544	0.010217	0.012105	0.014222	0.016585	0.019212	0.022119
43	0.005802	0.007094	0.008571	0.010247	0.012138	0.014259	0.016627	0.019258	0.022170
44	0.005822	0.007117	0.008597	0.010277	0.012172	0.014297	0.016669	0.019304	0.022221
45	0.005842	0.007140	0.008623	0.010307	0.012205	0.014334	0.016710	0.019350	0.022272
46	0.005862	0.007163	0.008650	0.010336	0.012239	0.014372	0.016752	0.019397	0.022324
47	0.005882	0.007186	0.008676	0.010366	0.012272	0.014409	0.016794	0.019443	0.022375
48	0.005903	0.007209	0.008702	0.010396	0.012306	0.014447	0.016836	0.019490	0.022426
49	0.005923	0.007233	0.008729	0.010426	0.012340	0.014485	0.016878	0.019536	0.022478
50	0.005943	0.007256	0.008756	0.010456	0.012373	0.014523	0.016920	0.019583	0.022529
51	0.005964	0.007280	0.008782	0.010486	0.012407	0.014560	0.016962	0.019630	0.022581
52	0.005984	0.007303	0.008809	0.010517	0.012441	0.014598	0.017004	0.019676	0.022633
53	0.006005	0.007327	0.008836	0.010547	0.012475	0.014636	0.017047	0.019723	0.022684
54	0.006025	0.007350	0.008863	0.010577	0.012509	0.014674	0.017089	0.019770	0.022736
55	0.006046	0.007374	0.008889	0.010608	0.012543	0.014713	0.017132	0.019817	0.022788
56	0.006067	0.007397	0.008916	0.010638	0.012578	0.014751	0.017174	0.019864	0.022840
57	0.006087	0.007421	0.008943	0.010669	0.012612	0.014789	0.017217	0.019912	0.022892
58	0.006108	0.007445	0.008970	0.010699	0.012646	0.014827	0.017259	0.019959	0.022944
59	0.006129	0.007469	0.008998	0.010730	0.012681	0.014866	0.017302	0.020006	0.022997
60	0.006150	0.007493	0.009025	0.010760	0.012715	0.014904	0.017345	0.020054	0.023049

Involute Functions for Angles from 23 to 32 Degrees

	Degrees								
	23	24	25	26	27	28	29	30	31
Minutes	Involute Functions								
0	0.023049	0.026350	0.029975	0.033947	0.038287	0.043017	0.048164	0.053752	0.059809
1	0.023102	0.026407	0.030039	0.034016	0.038362	0.043100	0.048253	0.053849	0.059914
2	0.023154	0.026465	0.030102	0.034086	0.038438	0.043182	0.048343	0.053946	0.060019
3	0.023207	0.026523	0.030166	0.034155	0.038514	0.043264	0.048432	0.054043	0.060124
4	0.023259	0.026581	0.030229	0.034225	0.038590	0.043347	0.048522	0.054140	0.060230
5	0.023312	0.026639	0.030293	0.034294	0.038666	0.043430	0.048612	0.054238	0.060335
6	0.023365	0.026697	0.030357	0.034364	0.038742	0.043513	0.048702	0.054336	0.060441
7	0.023418	0.026756	0.030420	0.034434	0.038818	0.043596	0.048792	0.054433	0.060547
8	0.023471	0.026814	0.030484	0.034504	0.038894	0.043679	0.048883	0.054531	0.060653
9	0.023524	0.026872	0.030549	0.034574	0.038971	0.043762	0.048973	0.054629	0.060759
10	0.023577	0.026931	0.030613	0.034644	0.039047	0.043845	0.049064	0.054728	0.060866
11	0.023631	0.026989	0.030677	0.034714	0.039124	0.043929	0.049154	0.054826	0.060972
12	0.023684	0.027048	0.030741	0.034785	0.039201	0.044012	0.049245	0.054924	0.061079
13	0.023738	0.027107	0.030806	0.034855	0.039278	0.044096	0.049336	0.055023	0.061186
14	0.023791	0.027166	0.030870	0.034926	0.039355	0.044180	0.049427	0.055122	0.061292
15	0.023845	0.027225	0.030935	0.034997	0.039432	0.044264	0.049518	0.055221	0.061400
16	0.023899	0.027284	0.031000	0.035067	0.039509	0.044348	0.049609	0.055320	0.061507
17	0.023952	0.027343	0.031065	0.035138	0.039586	0.044432	0.049701	0.055419	0.061614
18	0.024006	0.027402	0.031130	0.035209	0.039664	0.044516	0.049792	0.055518	0.061721
19	0.024060	0.027462	0.031195	0.035280	0.039741	0.044601	0.049884	0.055617	0.061829
20	0.024114	0.027521	0.031260	0.035352	0.039819	0.044685	0.049976	0.055717	0.061937
21	0.024169	0.027581	0.031325	0.035423	0.039897	0.044770	0.050068	0.055817	0.062045
22	0.024223	0.027640	0.031390	0.035494	0.039974	0.044855	0.050160	0.055916	0.062153
23	0.024277	0.027700	0.031456	0.035566	0.040052	0.044940	0.050252	0.056016	0.062261
24	0.024332	0.027760	0.031521	0.035637	0.040131	0.045024	0.050344	0.056116	0.062369
25	0.024386	0.027820	0.031587	0.035709	0.040209	0.045110	0.050437	0.056217	0.062478
26	0.024441	0.027880	0.031653	0.035781	0.040287	0.045195	0.050529	0.056317	0.062586
27	0.024495	0.027940	0.031718	0.035853	0.040366	0.045280	0.050622	0.056417	0.062695
28	0.024550	0.028000	0.031784	0.035925	0.040444	0.045366	0.050715	0.056518	0.062804
29	0.024605	0.028060	0.031850	0.035997	0.040523	0.045451	0.050808	0.056619	0.062913
30	0.024660	0.028121	0.031917	0.036069	0.040602	0.045537	0.050901	0.056720	0.063022
31	0.024715	0.028181	0.031983	0.036142	0.040680	0.045623	0.050994	0.056821	0.063131
32	0.024770	0.028242	0.032049	0.036214	0.040759	0.045709	0.051087	0.056922	0.063241
33	0.024825	0.028302	0.032116	0.036287	0.040839	0.045795	0.051181	0.057023	0.063350
34	0.024881	0.028363	0.032182	0.036359	0.040918	0.045881	0.051274	0.057124	0.063460
35	0.024936	0.028424	0.032249	0.036432	0.040997	0.045967	0.051368	0.057226	0.063570
36	0.024992	0.028485	0.032315	0.036505	0.041077	0.046054	0.051462	0.057328	0.063680
37	0.025047	0.028546	0.032382	0.036578	0.041156	0.046140	0.051556	0.057429	0.063790
38	0.025103	0.028607	0.032449	0.036651	0.041236	0.046227	0.051650	0.057531	0.063901
39	0.025159	0.028668	0.032516	0.036724	0.041316	0.046313	0.051744	0.057633	0.064011
40	0.025214	0.028729	0.032583	0.036798	0.041395	0.046400	0.051838	0.057736	0.064122
41	0.025270	0.028791	0.032651	0.036871	0.041475	0.046487	0.051933	0.057838	0.064232
42	0.025326	0.028852	0.032718	0.036945	0.041556	0.046575	0.052027	0.057940	0.064343
43	0.025382	0.028914	0.032785	0.037018	0.041636	0.046662	0.052122	0.058043	0.064454
44	0.025439	0.028976	0.032853	0.037092	0.041716	0.046749	0.052217	0.058146	0.064565
45	0.025495	0.029037	0.032920	0.037166	0.041797	0.046837	0.052312	0.058249	0.064677
46	0.025551	0.029099	0.032988	0.037240	0.041877	0.046924	0.052407	0.058352	0.064788
47	0.025608	0.029161	0.033056	0.037314	0.041958	0.047012	0.052502	0.058455	0.064900
48	0.025664	0.029223	0.033124	0.037388	0.042039	0.047100	0.052597	0.058558	0.065012
49	0.025721	0.029285	0.033192	0.037462	0.042120	0.047188	0.052693	0.058662	0.065123
50	0.025778	0.029348	0.033260	0.037537	0.042201	0.047276	0.052788	0.058765	0.065236
51	0.025834	0.029410	0.033328	0.037611	0.042282	0.047364	0.052884	0.058869	0.065348
52	0.025891	0.029472	0.033397	0.037686	0.042363	0.047452	0.052980	.058973	0.065460
53	0.025948	0.029535	0.033465	0.037761	0.042444	0.047541	0.053076	0.059077	0.065573
54	0.026005	0.029598	0.033534	0.037835	0.042526	0.047630	0.053172	0.059181	0.065685
55	0.026062	0.029660	0.033602	0.037910	0.042608	0.047718	0.053268	0.059285	0.065798
56	0.026120	0.029723	0.033671	0.037985	0.042689	0.047807	0.053365	0.059390	0.065911
57	0.026177	0.029786	0.033740	0.038060	0.042771	0.047896	0.053461	0.059494	0.066024
58	0.026235	0.029849	0.033809	0.038136	0.042853	0.047985	0.053558	0.059599	0.066137
59	0.026292	0.029912	0.033878	0.038211	0.042935	0.048074	0.053655	0.059704	0.066251
60	0.026350	0.029975	0.033947	0.038287	0.043017	0.048164	0.053752	0.059809	0.066364

Involute Functions for Angles from 32 to 41 Degrees

	Degrees								
	32	33	34	35	36	37	38	39	40
Minutes	Involute Functions								
0	0.066364	0.073449	0.081097	0.089342	0.098224	0.107782	0.118061	0.129106	0.140968
1	0.066478	0.073572	0.081229	0.089485	0.098378	0.107948	0.118238	0.129297	0.141173
2	0.066591	0.073695	0.081362	0.089628	0.098532	0.108113	0.118416	0.129488	0.141378
3	0.066705	0.073818	0.081494	0.089771	0.098686	0.108279	0.118594	0.129679	0.141584
4	0.066820	0.073941	0.081627	0.089914	0.098840	0.108445	0.118773	0.129870	0.141789
5	0.066934	0.074064	0.081760	0.090058	0.098994	0.108611	0.118951	0.130062	0.141995
6	0.067048	0.074188	0.081894	0.090201	0.099149	0.108777	0.119130	0.130254	0.142201
7	0.067163	0.074312	0.082027	0.090345	0.099303	0.108943	0.119309	0.130446	0.142408
8	0.067277	0.074435	0.082161	0.090489	0.099458	0.109110	0.119488	0.130639	0.142614
9	0.067392	0.074559	0.082294	0.090633	0.099614	0.109277	0.119667	0.130832	0.142821
10	0.067507	0.074684	0.082428	0.090777	0.099769	0.109444	0.119847	0.131025	0.143028
11	0.067622	0.074808	0.082562	0.090922	0.099924	0.109611	0.120027	0.131218	0.143236
12	0.067738	0.074932	0.082697	0.091067	0.100080	0.109779	0.120207	0.131411	0.143443
13	0.067853	0.075057	0.082831	0.091211	0.100236	0.109947	0.120387	0.131605	0.143651
14	0.067969	0.075182	0.082966	0.091356	0.100392	0.110114	0.120567	0.131799	0.143859
15	0.068084	0.075307	0.083101	0.091502	0.100549	0.110283	0.120748	0.131993	0.144068
16	0.068200	0.075432	0.083235	0.091647	0.100705	0.110451	0.120929	0.132187	0.144276
17	0.068316	0.075557	0.083371	0.091793	0.100862	0.110619	0.121110	0.132381	0.144485
18	0.068432	0.075683	0.083506	0.091938	0.101019	0.110788	0.121291	0.132576	0.144694
19	0.068549	0.075808	0.083641	0.092084	0.101176	0.110957	0.121473	0.132771	0.144903
20	0.068665	0.075934	0.083777	0.092230	0.101333	0.111126	0.121655	0.132966	0.145113
21	0.068782	0.076060	0.083913	0.092377	0.101490	0.111295	0.121837	0.133162	0.145323
22	0.068899	0.076186	0.084049	0.092523	0.101648	0.111465	0.122019	0.133358	0.145533
23	0.069016	0.076312	0.084185	0.092670	0.101806	0.111635	0.122201	0.133553	0.145743
24	0.069133	0.076439	0.084321	0.092816	0.101964	0.111805	0.122384	0.133750	0.145954
25	0.069250	0.076565	0.084458	0.092963	0.102122	0.111975	0.122567	0.133946	0.146165
26	0.069367	0.076692	0.084594	0.093111	0.102280	0.112145	0.122750	0.134143	0.146376
27	0.069485	0.076819	0.084731	0.093258	0.102439	0.112316	0.122933	0.134339	0.146587
28	0.069602	0.076946	0.084868	0.093406	0.102598	0.112486	0.123117	0.134537	0.146799
29	0.069720	0.077073	0.085005	0.093553	0.102757	0.112657	0.123300	0.134734	0.147010
30	0.069838	0.077200	0.085142	0.093701	0.102916	0.112829	0.123484	0.134931	0.147222
31	0.069956	0.077328	0.085280	0.093849	0.103075	0.113000	0.123668	0.135129	0.147435
32	0.070075	0.077455	0.085418	0.093998	0.103235	0.113172	0.123853	0.135327	0.147647
33	0.070193	0.077583	0.085555	0.094146	0.103395	0.113343	0.124037	0.135525	0.147860
34	0.070312	0.077711	0.085693	0.094295	0.103555	0.113515	0.124222	0.135724	0.148073
35	0.070430	0.077839	0.085832	0.094443	0.103715	0.113688	0.124407	0.135923	0.148286
36	0.070549	0.077968	0.085970	0.094593	0.103875	0.113860	0.124592	0.136122	0.148500
37	0.070668	0.078096	0.086108	0.094742	0.104036	0.114033	0.124778	0.136321	0.148714
38	0.070788	0.078225	0.086247	0.094891	0.104196	0.114205	0.124964	0.136520	0.148928
39	0.070907	0.078354	0.086386	0.095041	0.104357	0.114378	0.125150	0.136720	0.149142
40	0.071026	0.078483	0.086525	0.095190	0.104518	0.114552	0.125336	0.136920	0.149357
41	0.071146	0.078612	0.086664	0.095340	0.104680	0.114725	0.125522	0.137120	0.149572
42	0.071266	0.078741	0.086804	0.095490	0.104841	0.114899	0.125709	0.137320	0.149787
43	0.071386	0.078871	0.086943	0.095641	0.105003	0.115073	0.125896	0.137521	0.150002
44	0.071506	0.079000	0.087083	0.095791	0.105165	0.115247	0.126083	0.137722	0.150218
45	0.071626	0.079130	0.087223	0.095942	0.105327	0.115421	0.126270	0.137923	0.150434
46	0.071747	0.079260	0.087363	0.096093	0.105489	0.115595	0.126457	0.138124	0.150650
47	0.071867	0.079390	0.087503	0.096244	0.105652	0.115770	0.126645	0.138326	0.150866
48	0.071988	0.079520	0.087644	0.096395	0.105814	0.115945	0.126833	0.138528	0.151083
49	0.072109	0.079651	0.087784	0.096546	0.105977	0.116120	0.127021	0.138730	0.151299
50	0.072230	0.079781	0.087925	0.096698	0.106140	0.116296	0.127209	0.138932	0.151517
51	0.072351	0.079912	0.088066	0.096850	0.106304	0.116471	0.127398	0.139134	0.151734
52	0.072473	0.080043	0.088207	0.097002	0.106467	0.116647	0.127587	0.139337	0.151952
53	0.072594	0.080174	0.088348	0.097154	0.106631	0.116823	0.127776	0.139540	0.152169
54	0.072716	0.080306	0.088490	0.097306	0.106795	0.116999	0.127965	0.139743	0.152388
55	0.072838	0.080437	0.088631	0.097459	0.106959	0.117175	0.128155	0.139947	0.152606
56	0.072960	0.080569	0.088773	0.097611	0.107123	0.117352	0.128344	0.140151	0.152825
57	0.073082	0.080700	0.088915	0.097764	0.107288	0.117529	0.128534	0.140355	0.153044
58	0.073204	0.080832	0.089057	0.097917	0.107452	0.117706	0.128725	0.140559	0.153263
59	0.073326	0.080964	0.089200	0.098071	0.107617	0.117883	0.128915	0.140763	0.153482
60	0.073449	0.081097	0.089342	0.098224	0.107782	0.118061	0.129106	0.140968	0.153702

Involute Functions for Angles from 41 to 50 Degrees

	Degrees								
	41	42	43	44	45	46	47	48	49
Minutes	Involute Functions								
0	0.153702	0.167366	0.182024	0.197744	0.214602	0.232679	0.252064	0.272855	0.295157
1	0.153922	0.167602	0.182277	0.198015	0.214893	0.232991	0.252399	0.273214	0.295542
2	0.154142	0.167838	0.182530	0.198287	0.215184	0.233304	0.252734	0.273573	0.295928
3	0.154362	0.168075	0.182784	0.198559	0.215476	0.233616	0.253069	0.273933	0.296314
4	0.154583	0.168311	0.183038	0.198832	0.215768	0.233930	0.253405	0.274293	0.296701
5	0.154804	0.168548	0.183292	0.199104	0.216061	0.234243	0.253742	0.274654	0.297088
6	0.155025	0.168786	0.183547	0.199377	0.216353	0.234557	0.254078	0.275015	0.297475
7	0.155247	0.169023	0.183801	0.199651	0.216646	0.234871	0.254415	0.275376	0.297863
8	0.155469	0.169261	0.184057	0.199924	0.216940	0.235186	0.254753	0.275738	0.298251
9	0.155691	0.169500	0.184312	0.200198	0.217234	0.235501	0.255091	0.276101	0.298640
10	0.155913	0.169738	0.184568	0.200473	0.217528	0.235816	0.255429	0.276464	0.299029
11	0.156135	0.169977	0.184824	0.200747	0.217822	0.236132	0.255767	0.276827	0.299419
12	0.156358	0.170216	0.185080	0.201022	0.218117	0.236448	0.256106	0.277191	0.299809
13	0.156581	0.170455	0.185337	0.201297	0.218412	0.236765	0.256446	0.277555	0.300200
14	0.156805	0.170695	0.185594	0.201573	0.218708	0.237082	0.256786	0.277919	0.300591
15	0.157028	0.170935	0.185851	0.201849	0.219004	0.237399	0.257126	0.278284	0.300983
16	0.157252	0.171175	0.186109	0.202125	0.219300	0.237717	0.257467	0.278649	0.301375
17	0.157476	0.171415	0.186367	0.202401	0.219596	0.238035	0.257808	0.279015	0.301767
18	0.157701	0.171656	0.186625	0.202678	0.219893	0.238353	0.258149	0.279381	0.302160
19	0.157925	0.171897	0.186883	0.202956	0.220190	0.238672	0.258491	0.279748	0.302553
20	0.158150	0.172138	0.187142	0.203233	0.220488	0.238991	0.258833	0.280115	0.302947
21	0.158375	0.172380	0.187401	0.203511	0.220786	0.239310	0.259176	0.280483	0.303342
22	0.158601	0.172621	0.187661	0.203789	0.221084	0.239630	0.259519	0.280851	0.303736
23	0.158826	0.172864	0.187920	0.204067	0.221383	0.239950	0.259862	0.281219	0.304132
24	0.159052	0.173106	0.188180	0.204346	0.221682	0.240271	0.260206	0.281588	0.304527
25	0.159279	0.173349	0.188440	0.204625	0.221981	0.240592	0.260550	0.281957	0.304924
26	0.159505	0.173592	0.188701	0.204905	0.222281	0.240913	0.260895	0.282327	0.305320
27	0.159732	0.173835	0.188962	0.205185	0.222581	0.241235	0.261240	0.282697	0.305718
28	0.159959	0.174078	0.189223	0.205465	0.222881	0.241557	0.261585	0.283067	0.306115
29	0.160186	0.174322	0.189485	0.205745	0.223182	0.241879	0.261931	0.283438	0.306513
30	0.160414	0.174566	0.189746	0.206026	0.223483	0.242202	0.262277	0.283810	0.306912
31	0.160642	0.174811	0.190009	0.206307	0.223784	0.242525	0.262624	0.284182	0.307311
32	0.160870	0.175055	0.190271	0.206588	0.224086	0.242849	0.262971	0.284554	0.307710
33	0.161098	0.175300	0.190534	0.206870	0.224388	0.243173	0.263318	0.284927	0.308110
34	0.161327	0.175546	0.190797	0.207152	0.224690	0.243497	0.263666	0.285300	0.308511
35	0.161555	0.175791	0.191060	0.207434	0.224993	0.243822	0.264014	0.285673	0.308911
36	0.161785	0.176037	0.191324	0.207717	0.225296	0.244147	0.264363	0.286047	0.309313
37	0.162014	0.176283	0.191588	0.208000	0.225600	0.244472	0.264712	0.286422	0.309715
38	0.162244	0.176529	0.191852	0.208284	0.225904	0.244798	0.265062	0.286797	0.310117
39	0.162474	0.176776	0.192116	0.208567	0.226208	0.245125	0.265412	0.287172	0.310520
40	0.162704	0.177023	0.192381	0.208851	0.226512	0.245451	0.265762	0.287548	0.310923
41	0.162934	0.177270	0.192646	0.209136	0.226817	0.245778	0.266113	0.287924	0.311327
42	0.163165	0.177518	0.192912	0.209420	0.227123	0.246106	0.266464	0.288301	0.311731
43	0.163396	0.177766	0.193178	0.209705	0.227428	0.246433	0.266815	0.288678	0.312136
44	0.163628	0.178014	0.193444	0.209991	0.227734	0.246761	0.267167	0.289056	0.312541
45	0.163859	0.178262	0.193710	0.210276	0.228041	0.247090	0.267520	0.289434	0.312947
46	0.164091	0.178511	0.193977	0.210562	0.228347	0.247419	0.267872	0.289812	0.313353
47	0.164323	0.178760	0.194244	0.210849	0.228654	0.247748	0.268225	0.290191	0.313759
48	0.164556	0.179009	0.194511	0.211136	0.228962	0.248078	0.268579	0.290570	0.314166
49	0.164788	0.179259	0.194779	0.211423	0.229270	0.248408	0.268933	0.290950	0.314574
50	0.165021	0.179509	0.195047	0.211710	0.229578	0.248738	0.269287	0.291330	0.314982
51	0.165254	0.179759	0.195315	0.211998	0.229886	0.249069	0.269642	0.291711	0.315391
52	0.165488	0.180009	0.195584	0.212286	0.230195	0.249400	0.269998	0.292092	0.315800
53	0.165722	0.180260	0.195853	0.212574	0.230504	0.249732	0.270353	0.292474	0.316209
54	0.165956	0.180511	0.196122	0.212863	0.230814	0.250064	0.270709	0.292856	0.316619
55	0.166190	0.180763	0.196392	0.213152	0.231124	0.250396	0.271066	0.293238	0.317029
56	0.166425	0.181014	0.196661	0.213441	0.231434	0.250729	0.271423	0.293621	0.317440
57	0.166660	0.181266	0.196932	0.213731	0.231745	0.251062	0.271780	0.294004	0.317852
58	0.166895	0.181518	0.197202	0.214021	0.232056	0.251396	0.272138	0.294388	0.318264
59	0.167130	0.181771	0.197473	0.214311	0.232367	0.251730	0.272496	0.294772	0.318676
60	0.167366	0.182024	0.197744	0.214602	0.232679	0.252064	0.272855	0.295157	0.319089

LOGARITHMS

Logarithms are used to facilitate and shorten calculations involving multiplication, division, the extraction of roots, and obtaining powers of numbers. The following properties of logarithms are useful in solving problems of this type:

$$\log_c c = 1 \qquad \log_c c^p = p \qquad \log_c 1 = 0$$

$$\log_c(a \times b) = \log_c a + \log_c b \qquad \log_c(a \div b) = \log_c a - \log_c b$$

$$\log_c(a^p) = p\log_c a \qquad \log_c(\sqrt[p]{a}) = 1/p\log_c a$$

The logarithm of a number is defined as the exponent of a base number raised to a power. For example, $\log_{10} 3.162277 = 0.500$ means the logarithm of 3.162277 is equal to 0.500. Another way of expressing the same relationship is $10^{0.500} = 3.162277$, where 10 is the base number and the exponent 0.500 is the logarithm of 3.162277. A common example of a logarithmic expression $10^2 = 100$ means that the base 10 logarithm of 100 is 2, that is, $\log_{10} 100 = 2.00$. There are two standard systems of logarithms in use: the "common" system (base 10) and the so-called "natural" system (base $e = 2.71828+$). Logarithms to base e are frequently written using "ln" instead of "$\log_e$" such as ln 6.1 = 1.808289. Logarithms of a number can be converted between the natural- and common-based systems as follows: $\ln_e A = 2.3026 \times \log_{10} A$ and $\log_{10} A = 0.43430 \times \ln_e A$. Additional information on the use of "natural logarithms" is given at the end of this section.

A logarithm consists of two parts, a whole number and a decimal. The whole number, which may be positive, negative, or zero, is called the characteristic; the decimal is called the mantissa. As a rule, only the decimal or mantissa is given in tables of common logarithms; tables of natural logarithms give both the characteristic and mantissa. The tables given in this section are abbreviated, but very accurate results can be obtained by using the method of interpolation described in *Interpolation from the Tables* that follows. These tables are especially useful for finding logarithms and calculating powers and roots of numbers on calculators without these functions built in.

Evaluating Logarithms

Common Logarithms.—For common logarithms, the characteristic is prefixed to the mantissa according to the following rules: For numbers greater than or equal to 1, the characteristic is one less than the number of places to the left of the decimal point. For example, the characteristic of the logarithm of 237 is 2, and of 2536.5 is 3. For numbers smaller than 1 and greater than 0, the characteristic is negative and its numerical value is one more than the number of zeros immediately to the right of the decimal point. For example, the characteristic of the logarithm of 0.036 is –2, and the characteristic of the logarithm of 0.0006 is –4. The minus sign is usually written over the figure, as in $\bar{2}$ to indicate that the minus sign refers only to the characteristic and not to the mantissa, which is never negative. The logarithm of 0 does not exist.

The table of common logarithms in this section gives the mantissas of the logarithms of numbers from 1 to 10 and from 1.00 to 1.01. When finding the mantissa, the decimal point in a number is disregarded. The mantissa of the logarithms of 2716, 271.6, 27.16, 2.716, or 0.02716, for example, is the same. The tables give directly the mantissas of logarithms of numbers with three figures or less; the logarithms for numbers with four or more figures can be found by interpolation, as described in *Interpolation from the Tables* and illustrated in the examples. All the mantissas in the common logarithmic tables are decimals and the decimal point has been omitted in the table. However, a decimal point should always be put before the mantissa as soon as it is taken from the table. Logarithmic tables are sufficient for many purposes, but electronic calculators and computers are faster, simpler, and more accurate than tables.

To find the common logarithm of a number from the tables, find the left-hand column of the table and follow down to locate the first two figures of the number. Then look at the top row of the table, on the same page, and follow across it to find the third figure of the number. Follow down the column containing this last figure until opposite the row on which the first two figures were found. The number at the intersection of the row and column is the mantissa of the logarithm. If the logarithm of a number with less than three figures is being obtained, add extra zeros to the right of the number so as to obtain three figures. For example, if the mantissa of the logarithm of 6 is required, find the mantissa of 600.

Interpolation from the Tables.—If the logarithm of a number with more than three figures is needed, linear interpolation is a method of using two values from the table to estimate the value of the logarithm desired. To find the logarithm of a number not listed in the tables, find the mantissa corresponding to the first three digits of the given number (disregarding the decimal point and leading zeros) and find the mantissa of the first three digits of the given number plus one. For example, to find the logarithm of 601.2, 60.12, or 0.006012, find the mantissa of 601 and find the mantissa of 602 from the tables. Then subtract the mantissa of the smaller number from the mantissa of the larger number and multiply the result by a decimal number made from the remaining (additional greater than 3) figures of the original number. Add the result to the mantissa of the smaller number. Find the characteristic as described previously.

Example: Find the logarithm of 4032. The characteristic portion of the logarithm found in the manner described before is 3. Find the mantissa by locating 40 in the left-hand column of the logarithmic tables and then follow across the top row of the table to the column headed 3. Follow down the 3 column to the intersection with the 40 row and read the mantissa. The mantissa of the logarithm of 4030 is 0.605305. Because 4032 is between 4030 and 4040, the logarithm of 4032 is the logarithm of 4030 plus two tenths of the difference in the logarithms of 4030 and 4040. Find the mantissa of 4040 and then subtract from it the mantissa of 4030. Multiply the difference obtained by 0.2 and add the result to the mantissa of the logarithm of 4030. Finally, add the characteristic portion of the logarithm. The result is $\log_{10} 4032 = 3 + 0.605305 + 0.2 \times (0.606381 - 0.605305) = 3.60552$.

Finding a Number Whose Logarithm Is Given.—When a logarithm is given and it is required to find the corresponding number, find the number in the body of the table equal to the value of the mantissa. This value may appear in any column 0 to 9. Follow the row on which the mantissa is found across to the left to read the first two digits of the number sought. Read the third digit of the number from the top row of the table by following up the column on which the mantissa is found to the top. If the characteristic of the logarithm is positive, the number of figures to the left of the decimal in the number is one greater than the value of the characteristic. For example, if the figures corresponding to a given mantissa are 376 and the characteristic is 5, then the number sought has six figures to the left of the decimal point and is 376,000. If the characteristic had been $\overline{3}$, then the number sought would have been 0.00376. If the mantissa is not exactly obtainable in the tables, find the mantissa in the table that is nearest to the one given and determine the corresponding number. This procedure usually gives sufficiently accurate results. If more accuracy is required, find the two mantissas in the tables nearest to the mantissa given, one smaller and the other larger. For each of the two mantissas, read the three corresponding digits from the left column and top row to obtain the first three figures of the number as described before. The exact number sought lies between the two numbers found in this manner.

Next: 1) subtract the smaller mantissa from the given mantissa and; and 2) subtract the smaller mantissa from the larger mantissa.

Divide the result of (1) by the result of (2) and add the quotient to the number corresponding to the smaller mantissa.

Example: Find the number whose logarithm is 2.70053. First, find the number closest to the mantissa 70053 in the body of the tables. The closest mantissa listed in the tables is

700704, so read across the table to the left to find the first two digits of the number sought (50) and up the column to find the third digit of the number (2). The characteristic of the logarithm given is 2, so the number sought has three digits to the left of the decimal point. Therefore, the number sought is slightly less than 502 and greater than 501. If greater accuracy is required, find the two mantissas in the table closest to the given mantissa (699838 and 700704). Subtract the smaller mantissa from the mantissa of the given logarithm and divide the result by the smaller mantissa subtracted from the larger mantissa. Add the result to the number corresponding to the smaller mantissa. The resulting answer is 501 + (700530 – 699838) ÷ (700704 – 699838) = 501 + 0.79 = 501.79.

Avoiding the Use of Negative Characteristics.—As previously explained, the logarithm of any number less than 1 has a negative characteristic and a positive mantissa. In many computations, the use of logarithms having negative characteristics is troublesome and frequently a source of error. A simple way to avoid this difficulty is to convert each logarithm having a negative characteristic into an equivalent logarithm having a positive characteristic. This is done according to the following method, which is based on the principle that any number can be simultaneously added to and subtracted from the characteristic of a logarithm without changing its value. Thus: log 1 = 0.000000 = 10.000000 – 10; log 0.3 = $\bar{1}.47712$ = 9.47712 – 10; log 0.000478 = $\bar{4}.67943$ = 6.67943 – 10. Usually, 10 to 20 are added to and subtracted from the characteristic, but any convenient number may be so used.

Natural Logarithms.—In certain formulas and in some branches of mathematical analysis, use is made of logarithms (formerly also called Napierian or hyperbolic logarithms). As previously mentioned, the base of this system, e = 2.7182818284+, is the limit of certain mathematical series. The logarithm of a number A to the base e is usually written $\log_e A$ or $\ln A$. Tables of natural logarithms for numbers ranging from 1 to 10 and 1.00 to 1.01 are given in this Handbook after the table of common logarithms. To obtain natural logs of numbers less than 1 or greater than 10, proceed as in the following examples: $\log_e 0.239 = \log_e 2.39 - \log_e 10$; $\log_e 0.0239 = \log_e 2.39 - 2\log_e 10$; $\log_e 239 = \log_e 2.39 + 2\log_e 10$; $\log_e 2390 = \log_e 2.39 + 3\log_e 10$, etc.

Using Calculators to Find Logarithms.—Usually, using a scientific calculator is the quickest and most accurate method of finding logarithms and numbers corresponding to given logarithms. On most scientific calculators, the key labeled **log** is used to find common logarithms (base 10) and the key labeled **ln** is used for finding natural logarithms (base e). The keystrokes to find a logarithm will vary slightly from one calculator to another, so specific instructions are not given. To find the number corresponding to a given logarithm: use the key labeled $\mathbf{10^x}$ if a common logarithm is given or use the key labeled $\mathbf{e^x}$ if a natural logarithm is given; calculators without the 10^x or e^x keys may have a key labeled $\mathbf{x^y}$ that can be used by substituting 10 or e (2.718281 …), as required, for x and substituting the logarithm whose corresponding number is sought for y. On some other calculators, the **log** and **ln** keys are used to find common and natural logarithms, and the same keys in combination with the **INV**, or inverse, key are used to find the number corresponding to a given logarithm.

Multiplication by Logarithms.—If two or more numbers are to be multiplied together, find the logarithms of the numbers to be multiplied, and add these logarithms. The sum is the logarithm of the product, and the number corresponding to this logarithm, as found from the logarithmic tables, is the required product.

Example 1: Find the product of 2831 × 2.692 × 29.69 × 19.4

$$\begin{array}{llr}
\log 2831 & = 3.451786 + 0.1 \times (0.453318 - 0.451786) = & 3.451939 \\
\log 2.692 & = 0.429752 + 0.2 \times (0.431364 - 0.429752) = & 0.430074 \\
\log 29.69 & = 1.471292 + 0.9 \times (0.472756 - 0.471292) = & 1.472610 \\
\log 19.4 & = & \underline{1.287802} \\
 & & 6.642425
\end{array}$$

The closest number in the table corresponding to the mantissa is 439. The characteristic indicates the number has seven digits to the left of the decimal point; therefore, the product is slightly less than 4,390,000. If a more accurate result is required, interpolate from the table as follows: 438 + (0.642425 – 0.641474) ÷ (0.642465 – 0.641474) = 438.95963. Therefore, the product sought is 4,389,596.

In multiplication problems involving numbers less than 1, the method of avoiding the use of negative characteristics simplifies the addition and tends to reduce the possibility of error.

Example: Find the product of $0.002656 \times 155.1 \times 0.5833 \times 7.968$

$$
\begin{aligned}
\log\ 0.002656 &= \overline{3}.424228 = 7.424228 - 10\\
\log\ 155.1 &= 2.190611 = 2.190611\\
\log\ 0.5853 &= \overline{1}.767379 = 9.767379 - 10\\
\log\ 7.968 &= 0.901349 = \underline{0.901349}\\
& \qquad 20.283567 - 20 = 0.283567
\end{aligned}
$$

Therefore, the product is 1.92. Interpolate for additional accuracy if required.

Division by Logarithms.—When dividing one number by another, subtract the logarithm of the divisor from the logarithm of the dividend; the remainder is the logarithm of the quotient.

Example: Find the quotient of 7658 ÷ 935.3

$$
\begin{aligned}
\log\ 7658. &= 3.884115\\
-\log\ 935.3 &= \underline{-2.970951}\\
& \quad 0.913164
\end{aligned}
$$

From the tables, 818 + (0.913164 – 0.912753) ÷ (0.913284 – 0.912753) = 818.8. The answer has one digit to the left of the decimal; hence, 7658 ÷ 935.3 = 8.188.

Instead of dividing 7658 by 935.3, the same answer would be obtained if 7658 were multiplied by the reciprocal of 935.3, or 1 ÷ 935.3. To do this by logarithms, the log of 7658 and the log of the reciprocal of 935.3 would be added together.

To find the logarithm of the reciprocal of a number, subtract the log of the number from the log of 1. To do this conveniently, some number, such as 10, is first added to and then subtracted from the characteristic of the log of 1.

Example: Find the log of the reciprocal of 935.3

$$
\begin{aligned}
\log 1 &= 0.000000 = 10.000000 - 10\\
-\log 935.3 &= -2.970951 = \underline{-2.970951}\\
\log (1 \div 935.3) &= 7.029049 - 10\\
\log (1 \div 935.3) &= \overline{3}.029049\\
1 \div 935.3 &= 0.001069
\end{aligned}
$$

The quotient of 7658 ÷ 935.3 can be found by adding the log of 7658 and the log of the reciprocal of 935.3.

$$
\begin{aligned}
\log 7658 &= 3.884115 = 3.884115\\
\log (1 \div 935.3) &= \overline{3}.029049 = \underline{7.0299049 - 10}\\
&= 10.913164 - 10 = 0.91317
\end{aligned}
$$

Hence, 7658 ÷ 935.3 = 8.188.

As is readily seen, this method is more cumbersome than the direct method where there is only one factor each in the dividend and divisor. However, it greatly facilitates the solution of problems in division involving several factors in the dividend and the divisor. In such a problem, the logarithm of each factor of the dividend is added to the logarithm of the reciprocal of each factor of the divisor.

Example: Find the quotient of

$$\frac{0.0272 \times 27.1 \times 12.6}{2.371 \times 0.007}$$

$$\begin{array}{lllll}
\log 0.0272 & = \bar{2}.43457 & = 8.434569 & - 10 & \\
\log 27.1 & = 1.432969 & = 1.432969 & & \\
\log 12.6 & = 1.100371 & = 1.100371 & & \\
\log (1 \div 2.371) & & = 9.625069 & - 10 & \\
\log (1 \div 0.007) & & = \underline{2.154902} & & \\
 & & 22.74788 & - 20 & = 2.74788
\end{array}$$

The quotient is $559 + (0.74788 - 0.747412) \div (0.748188 - 0.747412) = 559.6$.

In division problems where the divisor is larger than the dividend, the subtraction of logarithms is facilitated if some number is added to and subtracted from the log of the dividend. (The is the same method used to convert a logarithm with a negative characteristic to an equivalent logarithm with a positive characteristic except that it serves to convert a logarithm with a positive characteristic to one with a larger positive characteristic, but having the same value.)

Example: Find the quotient of $43.2 \div 971.4$

$$\begin{array}{llllll}
\log\ 43.2 & = & 1.63584 & = 11.635484 & - 10 & \\
-\log\ 971.4 & = & -2.987397 & = \underline{-2.987397} & & \\
 & & & 8.648087 & - 10 & = \bar{2}.648087
\end{array}$$

Hence, the quotient of $43.2 \div 971.4$ is 0.044472.

Obtaining the Powers of Numbers.— A number may be raised to any power by simply multiplying the logarithm of the number by the exponent of the number. The product gives the logarithm of the value of the power.

Example 1: Find the value of 6.51^3

$$\log 6.51 = 0.81358$$
$$3 \times 0.81358 = 2.44074$$

The logarithm 2.44074 is the logarithm of 6.51^3. Hence, 6.51^3 equals the number corresponding to this logarithm, as found from the tables, or $6.51^3 = 275.9$.

Example 2: Find the value of $12^{1.29}$

$$\log 12 = 1.07918$$
$$1.29 \times 1.07918 = 1.39214$$

Hence, $12^{1.29} = 24.67$.

Raising a decimal to a decimal power presents a somewhat more difficult problem because of the negative characteristic of the logarithm and the fact that the logarithm must be multiplied by a decimal exponent. The method previously outlined for avoiding the use of negative characteristics is helpful here.

Example 3: Find the value of $0.0813^{0.46}$

$$\log 0.0813 = \bar{2}.91009 = 8.91009 - 10$$
$$\log 0.0813^{0.46} = 0.46 \times (8.91009 - 10) = 4.09864 - 4.6$$

Subtract and add 0.6 to make the characteristic a whole number:

$$\begin{array}{rrcl} & 4.09864 & - & 4.6 \\ & -0.6 & + & 0.6 \\ \hline \log 0.0813^{0.46} = & 3.49864 & - & 4 = \bar{1}.49864 \end{array}$$

Hence, $0.0813^{0.46} = 0.3152$.

Extracting Roots by Logarithms.— Roots of numbers, for example, $\sqrt[5]{37}$, can be extracted easily by means of logarithms. The small (^{5}) in the radical ($\sqrt{\ }$) of the root sign is called the index of the root. Any root of a number may be found by dividing its logarithm by the index of the root; the quotient is the logarithm of the root.

Example 1: Find $\sqrt[3]{276}$

$$\log 276 = 2.44091$$
$$2.44091 \div 3 = 0.81364$$

Hence, $\log \sqrt[3]{276} = 0.81364$ and $\sqrt[3]{276} = 6.511$

Example 2: Find $\sqrt[3]{0.67}$

$$\log 0.67 = \bar{1}.82607$$

Here it is not possible to divide directly, because there is a negative characteristic and a positive mantissa, another instance where the method of avoiding the use of negative characteristics, previously outlined, is helpful. The preferred procedure is to add and subtract some number to the characteristic that is evenly divisible by the index of the root. The root index is 3, so 9 can be added to and subtracted from the characteristic, and the resulting logarithm divided by 3.

$$\log 0.67 = \bar{1}.82607 = 8.82607 - 9$$
$$\log \sqrt[3]{0.67} = \frac{8.82607 - 9}{3} = 2.94202 - 3$$
$$\log \sqrt[3]{0.67} = 2.94202 - 3 = \bar{1}.94202$$

Hence, $\sqrt[3]{0.67} = 0.875$

Example 3: Find $\sqrt[1.7]{0.2}$

$$\log 0.2 = \bar{1}.30103 = 16.30103 - 17$$
$$\log \sqrt[1.7]{0.2} = \frac{16.30103 - 17}{1.7} = 9.58884 - 10 = \bar{1}.58884$$

Hence,

$$\sqrt[1.7]{0.2} = 0.388$$

Table of Logarithms

Table of Common Logarithms

	0	1	2	3	4	5	6	7	8	9
10	000000	004321	008600	012837	017033	021189	025306	029384	033424	037426
11	041393	045323	049218	053078	056905	060698	064458	068186	071882	075547
12	079181	082785	086360	089905	093422	096910	100371	103804	107210	110590
13	113943	117271	120574	123852	127105	130334	133539	136721	139879	143015
14	146128	149219	152288	155336	158362	161368	164353	167317	170262	173186
15	176091	178977	181844	184691	187521	190332	193125	195900	198657	201397
16	204120	206826	209515	212188	214844	217484	220108	222716	225309	227887
17	230449	232996	235528	238046	240549	243038	245513	247973	250420	252853
18	255273	257679	260071	262451	264818	267172	269513	271842	274158	276462
19	278754	281033	283301	285557	287802	290035	292256	294466	296665	298853
20	301030	303196	305351	307496	309630	311754	313867	315970	318063	320146
21	322219	324282	326336	328380	330414	332438	334454	336460	338456	340444
22	342423	344392	346353	348305	350248	352183	354108	356026	357935	359835
23	361728	363612	365488	367356	369216	371068	372912	374748	376577	378398
24	380211	382017	383815	385606	387390	389166	390935	392697	394452	396199
25	397940	399674	401401	403121	404834	406540	408240	409933	411620	413300
26	414973	416641	418301	419956	421604	423246	424882	426511	428135	429752
27	431364	432969	434569	436163	437751	439333	440909	442480	444045	445604
28	447158	448706	450249	451786	453318	454845	456366	457882	459392	460898
29	462398	463893	465383	466868	468347	469822	471292	472756	474216	475671
30	477121	478566	480007	481443	482874	484300	485721	487138	488551	489958
31	491362	492760	494155	495544	496930	498311	499687	501059	502427	503791
32	505150	506505	507856	509203	510545	511883	513218	514548	515874	517196
33	518514	519828	521138	522444	523746	525045	526339	527630	528917	530200
34	531479	532754	534026	535294	536558	537819	539076	540329	541579	542825
35	544068	545307	546543	547775	549003	550228	551450	552668	553883	555094
36	556303	557507	558709	559907	561101	562293	563481	564666	565848	567026
37	568202	569374	570543	571709	572872	574031	575188	576341	577492	578639
38	579784	580925	582063	583199	584331	585461	586587	587711	588832	589950
39	591065	592177	593286	594393	595496	596597	597695	598791	599883	600973
40	602060	603144	604226	605305	606381	607455	608526	609594	610660	611723
41	612784	613842	614897	615950	617000	618048	619093	620136	621176	622214
42	623249	624282	625312	626340	627366	628389	629410	630428	631444	632457
43	633468	634477	635484	636488	637490	638489	639486	640481	641474	642465
44	643453	644439	645422	646404	647383	648360	649335	650308	651278	652246
45	653213	654177	655138	656098	657056	658011	658965	659916	660865	661813
46	662758	663701	664642	665581	666518	667453	668386	669317	670246	671173
47	672098	673021	673942	674861	675778	676694	677607	678518	679428	680336
48	681241	682145	683047	683947	684845	685742	686636	687529	688420	689309
49	690196	691081	691965	692847	693727	694605	695482	696356	697229	698101
50	698970	699838	700704	701568	702431	703291	704151	705008	705864	706718
51	707570	708421	709270	710117	710963	711807	712650	713491	714330	715167
52	716003	716838	717671	718502	719331	720159	720986	721811	722634	723456
53	724276	725095	725912	726727	727541	728354	729165	729974	730782	731589
54	732394	733197	733999	734800	735599	736397	737193	737987	738781	739572
55	740363	741152	741939	742725	743510	744293	745075	745855	746634	747412
56	748188	748963	749736	750508	751279	752048	752816	753583	754348	755112
57	755875	756636	757396	758155	758912	759668	760422	761176	761928	762679
58	763428	764176	764923	765669	766413	767156	767898	768638	769377	770115
59	770852	771587	772322	773055	773786	774517	775246	775974	776701	777427

Table of Common Logarithms

	0	1	2	3	4	5	6	7	8	9
60	778151	778874	779596	780317	781037	781755	782473	783189	783904	784617
61	785330	786041	786751	787460	788168	788875	789581	790285	790988	791691
62	792392	793092	793790	794488	795185	795880	796574	797268	797960	798651
63	799341	800029	800717	801404	802089	802774	803457	804139	804821	805501
64	806180	806858	807535	808211	808886	809560	810233	810904	811575	812245
65	812913	813581	814248	814913	815578	816241	816904	817565	818226	818885
66	819544	820201	820858	821514	822168	822822	823474	824126	824776	825426
67	826075	826723	827369	828015	828660	829304	829947	830589	831230	831870
68	832509	833147	833784	834421	835056	835691	836324	836957	837588	838219
69	838849	839478	840106	840733	841359	841985	842609	843233	843855	844477
70	845098	845718	846337	846955	847573	848189	848805	849419	850033	850646
71	851258	851870	852480	853090	853698	854306	854913	855519	856124	856729
72	857332	857935	858537	859138	859739	860338	860937	861534	862131	862728
73	863323	863917	864511	865104	865696	866287	866878	867467	868056	868644
74	869232	869818	870404	870989	871573	872156	872739	873321	873902	874482
75	875061	875640	876218	876795	877371	877947	878522	879096	879669	880242
76	880814	881385	881955	882525	883093	883661	884229	884795	885361	885926
77	886491	887054	887617	888179	888741	889302	889862	890421	890980	891537
78	892095	892651	893207	893762	894316	894870	895423	895975	896526	897077
79	897627	898176	898725	899273	899821	900367	900913	901458	902003	902547
80	903090	903633	904174	904716	905256	905796	906335	906874	907411	907949
81	908485	909021	909556	910091	910624	911158	911690	912222	912753	913284
82	913814	914343	914872	915400	915927	916454	916980	917506	918030	918555
83	919078	919601	920123	920645	921166	921686	922206	922725	923244	923762
84	924279	924796	925312	925828	926342	926857	927370	927883	928396	928908
85	929419	929930	930440	930949	931458	931966	932474	932981	933487	933993
86	934498	935003	935507	936011	936514	937016	937518	938019	938520	939020
87	939519	940018	940516	941014	941511	942008	942504	943000	943495	943989
88	944483	944976	945469	945961	946452	946943	947434	947924	948413	948902
89	949390	949878	950365	950851	951338	951823	952308	952792	953276	953760
90	954243	954725	955207	955688	956168	956649	957128	957607	958086	958564
91	959041	959518	959995	960471	960946	961421	961895	962369	962843	963316
92	963788	964260	964731	965202	965672	966142	966611	967080	967548	968016
93	968483	968950	969416	969882	970347	970812	971276	971740	972203	972666
94	973128	973590	974051	974512	974972	975432	975891	976350	976808	977266
95	977724	978181	978637	979093	979548	980003	980458	980912	981366	981819
96	982271	982723	983175	983626	984077	984527	984977	985426	985875	986324
97	986772	987219	987666	988113	988559	989005	989450	989895	990339	990783
98	991226	991669	992111	992554	992995	993436	993877	994317	994757	995196
99	995635	996074	996512	996949	997386	997823	998259	998695	999131	999565
100	000000	000434	000868	001301	001734	002166	002598	003029	003461	003891
101	004321	004751	005181	005609	006038	006466	006894	007321	007748	008174
102	008600	009026	009451	009876	010300	010724	011147	011570	011993	012415
103	012837	013259	013680	014100	014521	014940	015360	015779	016197	016616
104	017033	017451	017868	018284	018700	019116	019532	019947	020361	020775
105	021189	021603	022016	022428	022841	023252	023664	024075	024486	024896
106	025306	025715	026125	026533	026942	027350	027757	028164	028571	028978
107	029384	029789	030195	030600	031004	031408	031812	032216	032619	033021
108	033424	033826	034227	034628	035029	035430	035830	036230	036629	037028
109	037426	037825	038223	038620	039017	039414	039811	040207	040602	040998

Table of Natural Logarithms

	0	1	2	3	4	5	6	7	8	9
1.0	0.00000	0.009950	0.019803	0.029559	0.039221	0.048790	0.058269	0.067659	0.076961	0.086178
1.1	0.09531	0.104360	0.113329	0.122218	0.131028	0.139762	0.148420	0.157004	0.165514	0.173953
1.2	0.18232	0.190620	0.198851	0.207014	0.215111	0.223144	0.231112	0.239017	0.246860	0.254642
1.3	0.26236	0.270027	0.277632	0.285179	0.292670	0.300105	0.307485	0.314811	0.322083	0.329304
1.4	0.33647	0.343590	0.350657	0.357674	0.364643	0.371564	0.378436	0.385262	0.392042	0.398776
1.5	0.40546	0.412110	0.418710	0.425268	0.431782	0.438255	0.444686	0.451076	0.457425	0.463734
1.6	0.47000	0.476234	0.482426	0.488580	0.494696	0.500775	0.506818	0.512824	0.518794	0.524729
1.7	0.53062	0.536493	0.542324	0.548121	0.553885	0.559616	0.565314	0.570980	0.576613	0.582216
1.8	0.58778	0.593327	0.598837	0.604316	0.609766	0.615186	0.620576	0.625938	0.631272	0.636577
1.9	0.64185	0.647103	0.652325	0.657520	0.662688	0.667829	0.672944	0.678034	0.683097	0.688135
2.0	0.69314	0.698135	0.703098	0.708036	0.712950	0.717840	0.722706	0.727549	0.732368	0.737164
2.1	0.74193	0.746688	0.751416	0.756122	0.760806	0.765468	0.770108	0.774727	0.779325	0.783902
2.2	0.78845	0.792993	0.797507	0.802002	0.806476	0.810930	0.815365	0.819780	0.824175	0.828552
2.3	0.83290	0.837248	0.841567	0.845868	0.850151	0.854415	0.858662	0.862890	0.867100	0.871293
2.4	0.87546	0.879627	0.883768	0.887891	0.891998	0.896088	0.900161	0.904218	0.908259	0.912283
2.5	0.91629	0.920283	0.924259	0.928219	0.932164	0.936093	0.940007	0.943906	0.947789	0.951658
2.6	0.95551	0.959350	0.963174	0.966984	0.970779	0.974560	0.978326	0.982078	0.985817	0.989541
2.7	0.99325	0.996949	1.000632	1.004302	1.007958	1.011601	1.015231	1.018847	1.022451	1.026042
2.8	1.02961	1.033184	1.036737	1.040277	1.043804	1.047319	1.050822	1.054312	1.057790	1.061257
2.9	1.06471	1.068153	1.071584	1.075002	1.078410	1.081805	1.085189	1.088562	1.091923	1.095273
3.0	1.09861	1.101940	1.105257	1.108563	1.111858	1.115142	1.118415	1.121678	1.124930	1.128171
3.1	1.13140	1.134623	1.137833	1.141033	1.144223	1.147402	1.150572	1.153732	1.156881	1.160021
3.2	1.16315	1.166271	1.169381	1.172482	1.175573	1.178655	1.181727	1.184790	1.187843	1.190888
3.3	1.19392	1.196948	1.199965	1.202972	1.205971	1.208960	1.211941	1.214913	1.217876	1.220830
3.4	1.22377	1.226712	1.229641	1.232560	1.235471	1.238374	1.241269	1.244155	1.247032	1.249902
3.5	1.25276	1.255616	1.258461	1.261298	1.264127	1.266948	1.269761	1.272566	1.275363	1.278152
3.6	1.28093	1.283708	1.286474	1.289233	1.291984	1.294727	1.297463	1.300192	1.302913	1.305626
3.7	1.30833	1.311032	1.313724	1.316408	1.319086	1.321756	1.324419	1.327075	1.329724	1.332366
3.8	1.33500	1.337629	1.340250	1.342865	1.345472	1.348073	1.350667	1.353255	1.355835	1.358409
3.9	1.36097	1.363537	1.366092	1.368639	1.371181	1.373716	1.376244	1.378766	1.381282	1.383791
4.0	1.38629	1.388791	1.391282	1.393766	1.396245	1.398717	1.401183	1.403643	1.406097	1.408545
4.1	1.41098	1.413423	1.415853	1.418277	1.420696	1.423108	1.425515	1.427916	1.430311	1.432701
4.2	1.43508	1.437463	1.439835	1.442202	1.444563	1.446919	1.449269	1.451614	1.453953	1.456287
4.3	1.45861	1.460938	1.463255	1.465568	1.467874	1.470176	1.472472	1.474763	1.477049	1.479329
4.4	1.48160	1.483875	1.486140	1.488400	1.490654	1.492904	1.495149	1.497388	1.499623	1.501853
4.5	1.50407	1.506297	1.508512	1.510722	1.512927	1.515127	1.517323	1.519513	1.521699	1.523880
4.6	1.52605	1.528228	1.530395	1.532557	1.534714	1.536867	1.539015	1.541159	1.543298	1.545433
4.7	1.54756	1.549688	1.551809	1.553925	1.556037	1.558145	1.560248	1.562346	1.564441	1.566530
4.8	1.56861	1.570697	1.572774	1.574846	1.576915	1.578979	1.581038	1.583094	1.585145	1.587192
4.9	1.58923	1.591274	1.593309	1.595339	1.597365	1.599388	1.601406	1.603420	1.605430	1.607436
5.0	1.60943	1.611436	1.613430	1.615420	1.617406	1.619388	1.621366	1.623341	1.625311	1.627278
5.1	1.62924	1.631199	1.633154	1.635106	1.637053	1.638997	1.640937	1.642873	1.644805	1.646734
5.2	1.64865	1.650580	1.652497	1.654411	1.656321	1.658228	1.660131	1.662030	1.663926	1.665818
5.3	1.66770	1.669592	1.671473	1.673351	1.675226	1.677097	1.678964	1.680828	1.682688	1.684545
5.4	1.68639	1.688249	1.690096	1.691939	1.693779	1.695616	1.697449	1.699279	1.701105	1.702928
5.5	1.70474	1.706565	1.708378	1.710188	1.711995	1.713798	1.715598	1.717395	1.719189	1.720979
5.6	1.722767	1.724551	1.726332	1.728109	1.729884	1.731656	1.733424	1.735189	1.736951	1.738710
5.7	1.74046	1.742219	1.743969	1.745716	1.747459	1.749200	1.750937	1.752672	1.754404	1.756132
5.8	1.75785	1.759581	1.761300	1.763017	1.764731	1.766442	1.768150	1.769855	1.771557	1.773256
5.9	1.77495	1.776646	1.778336	1.780024	1.781709	1.783391	1.785070	1.786747	1.788421	1.790091

Table of Natural Logarithms

	0	1	2	3	4	5	6	7	8	9
6.0	1.791759	1.793425	1.795087	1.796747	1.798404	1.800058	1.801710	1.803359	1.805005	1.806648
6.1	1.808289	1.809927	1.811562	1.813195	1.814825	1.816452	1.818077	1.819699	1.821318	1.822935
6.2	1.824549	1.826161	1.827770	1.829376	1.830980	1.832581	1.834180	1.835776	1.837370	1.838961
6.3	1.840550	1.842136	1.843719	1.845300	1.846879	1.848455	1.850028	1.851599	1.853168	1.854734
6.4	1.856298	1.857859	1.859418	1.860975	1.862529	1.864080	1.865629	1.867176	1.868721	1.870263
6.5	1.871802	1.873339	1.874874	1.876407	1.877937	1.879465	1.880991	1.882514	1.884035	1.885553
6.6	1.887070	1.888584	1.890095	1.891605	1.893112	1.894617	1.896119	1.897620	1.899118	1.900614
6.7	1.902108	1.903599	1.905088	1.906575	1.908060	1.909543	1.911023	1.912501	1.913977	1.915451
6.8	1.916923	1.918392	1.919859	1.921325	1.922788	1.924249	1.925707	1.927164	1.928619	1.930071
6.9	1.931521	1.932970	1.934416	1.935860	1.937302	1.938742	1.940179	1.941615	1.943049	1.944481
7.0	1.945910	1.947338	1.948763	1.950187	1.951608	1.953028	1.954445	1.955860	1.957274	1.958685
7.1	1.960095	1.961502	1.962908	1.964311	1.965713	1.967112	1.968510	1.969906	1.971299	1.972691
7.2	1.974081	1.975469	1.976855	1.978239	1.979621	1.981001	1.982380	1.983756	1.985131	1.986504
7.3	1.987874	1.989243	1.990610	1.991976	1.993339	1.994700	1.996060	1.997418	1.998774	2.000128
7.4	2.001480	2.002830	2.004179	2.005526	2.006871	2.008214	2.009555	2.010895	2.012233	2.013569
7.5	2.014903	2.016235	2.017566	2.018895	2.020222	2.021548	2.022871	2.024193	2.025513	2.026832
7.6	2.028148	2.029463	2.030776	2.032088	2.033398	2.034706	2.036012	2.037317	2.038620	2.039921
7.7	2.041220	2.042518	2.043814	2.045109	2.046402	2.047693	2.048982	2.050270	2.051556	2.052841
7.8	2.054124	2.055405	2.056685	2.057963	2.059239	2.060514	2.061787	2.063058	2.064328	2.065596
7.9	2.066863	2.068128	2.069391	2.070653	2.071913	2.073172	2.074429	2.075684	2.076938	2.078191
8.0	2.079442	2.080691	2.081938	2.083185	2.084429	2.085672	2.086914	2.088153	2.089392	2.090629
8.1	2.091864	2.093098	2.094330	2.095561	2.096790	2.098018	2.099244	2.100469	2.101692	2.102914
8.2	2.104134	2.105353	2.106570	2.107786	2.109000	2.110213	2.111425	2.112635	2.113843	2.115050
8.3	2.116256	2.117460	2.118662	2.119863	2.121063	2.122262	2.123458	2.124654	2.125848	2.127041
8.4	2.128232	2.129421	2.130610	2.131797	2.132982	2.134166	2.135349	2.136531	2.137710	2.138889
8.5	2.140066	2.141242	2.142416	2.143589	2.144761	2.145931	2.147100	2.148268	2.149434	2.150599
8.6	2.151762	2.152924	2.154085	2.155245	2.156403	2.157559	2.158715	2.159869	2.161022	2.162173
8.7	2.163323	2.164472	2.165619	2.166765	2.167910	2.169054	2.170196	2.171337	2.172476	2.173615
8.8	2.174752	2.175887	2.177022	2.178155	2.179287	2.180417	2.181547	2.182675	2.183802	2.184927
8.9	2.186051	2.187174	2.188296	2.189416	2.190536	2.191654	2.192770	2.193886	2.195000	2.196113
9.0	2.197225	2.198335	2.199444	2.200552	2.201659	2.202765	2.203869	2.204972	2.206074	2.207175
9.1	2.208274	2.209373	2.210470	2.211566	2.212660	2.213754	2.214846	2.215937	2.217027	2.218116
9.2	2.219203	2.220290	2.221375	2.222459	2.223542	2.224624	2.225704	2.226783	2.227862	2.228939
9.3	2.230014	2.231089	2.232163	2.233235	2.234306	2.235376	2.236445	2.237513	2.238580	2.239645
9.4	2.240710	2.241773	2.242835	2.243896	2.244956	2.246015	2.247072	2.248129	2.249184	2.250239
9.5	2.251292	2.252344	2.253395	2.254445	2.255493	2.256541	2.257588	2.258633	2.259678	2.260721
9.6	2.261763	2.262804	2.263844	2.264883	2.265921	2.266958	2.267994	2.269028	2.270062	2.271094
9.7	2.272126	2.273156	2.274186	2.275214	2.276241	2.277267	2.278292	2.279316	2.280339	2.281361
9.8	2.282382	2.283402	2.284421	2.285439	2.286456	2.287471	2.288486	2.289500	2.290513	2.291524
9.9	2.292535	2.293544	2.294553	2.295560	2.296567	2.297573	2.298577	2.299581	2.300583	2.301585
1.00	0.000000	0.001000	0.001998	0.002996	0.003992	0.004988	0.005982	0.006976	0.007968	0.008960
1.01	0.009950	0.010940	0.011929	0.012916	0.013903	0.014889	0.015873	0.016857	0.017840	0.018822
1.02	0.019803	0.020783	0.021761	0.022739	0.023717	0.024693	0.025668	0.026642	0.027615	0.028587
1.03	0.029559	0.030529	0.031499	0.032467	0.033435	0.034401	0.035367	0.036332	0.037296	0.038259
1.04	0.039221	0.040182	0.041142	0.042101	0.043059	0.044017	0.044973	0.045929	0.046884	0.047837
1.05	0.048790	0.049742	0.050693	0.051643	0.052592	0.053541	0.054488	0.055435	0.056380	0.057325
1.06	0.058269	0.059212	0.060154	0.061095	0.062035	0.062975	0.063913	0.064851	0.065788	0.066724
1.07	0.067659	0.068593	0.069526	0.070458	0.071390	0.072321	0.073250	0.074179	0.075107	0.076035
1.08	0.076961	0.077887	0.078811	0.079735	0.080658	0.081580	0.082501	0.083422	0.084341	0.085260
1.09	0.086178	0.087095	0.088011	0.088926	0.089841	0.090754	0.091667	0.092579	0.093490	0.094401