

# Representing a Circle or a Sphere with NURBS

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This is just a brief note on representing circles and spheres with NURBS. For more information about NURBS, a good engineering-style approach to NURBS is [1]. A more mathematically advanced presentation is [2].

## 1 Representing a Circle

A quadrant of a circle can be represented as a NURBS curve of degree 2. The curve is  $x^2 + y^2 = 1$ ,  $x \geq 0$ ,  $y \geq 0$ . The general parameterization is

$$(x(u), y(u)) = \frac{w_0(1-u)^2(1,0) + w_1 2u(1-u)(1,1) + w_2 u^2(0,1)}{w_0(1-u)^2 + w_1 2u(1-u) + w_2 u^2}$$

for  $u \in [0, 1]$ . The requirement that  $x^2 + y^2 = 1$  leads to the weights constraint  $2w_1^2 = w_0 w_2$ . The choice of weights  $w_0 = 1$ ,  $w_1 = 1$ , and  $w_2 = 2$  leads to a commonly mentioned parameterization

$$(x(u), y(u)) = \frac{(1-u^2, 2u)}{1+u^2}$$

If you were to tessellate the curve with an odd number of uniform samples of  $u$ , say  $u_i = i/(2n)$  for  $0 \leq i \leq 2n$ , then the resulting polyline is not symmetric about the midpoint  $u = 1/2$ . To obtain a symmetric tessellation you need to choose  $w_0 = w_2$ . The weight constraint then implies  $w_0 = w_1 \sqrt{2}$ . The parameterization is then

$$(x(u), y(u)) = \frac{(\sqrt{2}(1-u)^2 + 2u(1-u), 2u(1-u) + \sqrt{2}u^2)}{\sqrt{2}(1-u)^2 + 2u(1-u) + \sqrt{2}u^2}$$

In either case we have a ratio of quadratic polynomials.

An algebraic construction of the same type, but quite a bit more tedious to solve, produces a ratio of quartic polynomials. The control points and control weights are required to be symmetric to obtain a tessellation that is symmetric about its midpoint. The middle weight is chosen as  $w_2 = 4$ .

$$(x(u), y(u)) = \frac{(1-u)^4 w_0(1,0) + 4(1-u)^3 u w_1(x_1, y_1) + 24(1-u)^2 u^2(x_2, x_2) + 4(1-u)u^3 w_1(y_1, x_1) + u^4 w_0}{(1-u)^4 w_0 + 4(1-u)^3 u w_1 + 24(1-u)^2 u^2 + 4(1-u)u^3 w_1 + u^4 w_0}$$

The parameters are  $x_1 = 1$ ,  $y_1 = (\sqrt{3}-1)/\sqrt{3}$ ,  $x_2 = (\sqrt{3}+1)/(2\sqrt{3})$ ,  $w_0 = 4\sqrt{3}(\sqrt{3}-1)$ , and  $w_1 = 3/\sqrt{2}$ .

## 2 Representing a Sphere

An octant of a sphere can be represented as a triangular NURBS surface patch of degree 4. A simple parameterization of  $x^2 + y^2 + z^2 = 1$  can be made by setting  $r^2 = x^2 + y^2$ . The sphere is then  $r^2 + z^2 = 1$ . Now apply the parameterization for a circle,

$$(r, z) = \frac{(1-u^2, 2u)}{1+u^2}$$

But  $(x/r)^2 + (y/r)^2 = 1$ , so another application of the parameterization for a circle is

$$\frac{(x, y)}{r} = \frac{(1-v^2, 2v)}{1+v^2}$$

Combining these produces

$$(x(u, v), y(u, v), z(u, v)) = \frac{((1 - u^2)(1 - v^2), (1 - u^2)2v, 2u(1 + v^2))}{(1 + u^2)(1 + v^2)}$$

The components are ratios of quartic polynomials. The domain is  $u \geq 0$ ,  $v \geq 0$ , and  $u + v \leq 1$ . In barycentric coordinates, set  $w = 1 - u - v$  so that  $u + v + w = 1$  with  $u$ ,  $v$ , and  $w$  nonnegative. In this setting, you can think of the octant of the sphere as a mapping from the  $uvw$ -triangle with vertices  $(1, 0, 0)$ ,  $(0, 1, 0)$ , and  $(0, 0, 1)$ . Although a valid parameterization, a symmetric subdivision of the the  $uvw$ -triangle does not lead to a symmetric tessellation of the sphere.

Another parameterization is provided in [2]. This one chooses symmetric control points and symmetric weights,

$$(x(u, v), y(u, v), z(u, v)) = \frac{\sum_{i=0}^4 \sum_{j=0}^{4-i} w_{i,j,4-i-j} \mathbf{P}_{i,j,4-i-j} B_{i,j}(u, v)}{\sum_{i=0}^4 \sum_{j=0}^{4-i} w_{i,j,4-i-j} B_{i,j}(u, v)}$$

where

$$B_{i,j}(u, v) = \frac{4!}{i!j!(4-i-j)!} u^i v^j (1-u-v)^{4-i-j}, \quad u \geq 0, \quad v \geq 0, \quad u + v \leq 1$$

are the Bernstein polynomials. The control points  $\mathbf{P}_{i,j,k}$  are defined in terms of three constants  $a_0 = (\sqrt{3} - 1)/\sqrt{3}$ ,  $a_1 = (\sqrt{3} + 1)/(2\sqrt{3})$ , and  $a_2 = 1 - (5 - \sqrt{2})(7 - \sqrt{3})/46$ ,

$$\begin{array}{llllll} \mathbf{P}_{040} & & & & & (0, 1, 0) \\ \mathbf{P}_{031} & \mathbf{P}_{130} & & & & (0, 1, a_0) \quad (a_0, 1, 0) \\ \mathbf{P}_{022} & \mathbf{P}_{121} & \mathbf{P}_{220} & = & (0, a_1, a_1) & (a_2, 1, a_2) \quad (a_1, a_1, 0) \\ \mathbf{P}_{013} & \mathbf{P}_{112} & \mathbf{P}_{211} & \mathbf{P}_{310} & (0, a_0, 1) & (a_2, a_2, 1) \quad (1, a_2, a_2) \quad (1, a_0, 0) \\ \mathbf{P}_{004} & \mathbf{P}_{103} & \mathbf{P}_{202} & \mathbf{P}_{301} & \mathbf{P}_{400} & (0, 0, 1) \quad (a_0, 0, 1) \quad (a_1, 0, a_1) \quad (1, 0, a_0) \quad (1, 0, 0) \end{array}$$

The control weights  $w_{i,j,k}$  are defined in terms of four constants,  $b_0 = 4\sqrt{3}(\sqrt{3} - 1)$ ,  $b_1 = 3\sqrt{2}$ ,  $b_2 = 4$ , and  $b_3 = \sqrt{2}(3 + 2\sqrt{2} - \sqrt{3})/\sqrt{3}$ ,

$$\begin{array}{llllllll} w_{040} & & & & & & & & b_0 \\ w_{031} & w_{130} & & & & & & & b_1 \quad b_1 \\ w_{022} & w_{121} & w_{220} & & & = & b_2 & b_3 & b_2 \\ w_{013} & w_{112} & w_{211} & w_{310} & & & b_1 & b_3 & b_3 & b_1 \\ w_{004} & w_{103} & w_{202} & w_{301} & w_{400} & & b_0 & b_1 & b_2 & b_1 & b_0 \end{array}$$

Both the numerator and denominator polynomial are quartic polynomials. Notice that each boundary curve of the triangle patch is a quartic polynomial of one variable that is exactly what was shown earlier for a quadrant of a circle.

## References

- [1] David F. Rogers, *An Introduction to NURBS with Historical Perspective*, Morgan Kaufmann Publishers, San Francisco, CA, 2001

- [2] *Curves and Surfaces for Computer Aided Geometric Design: A Practical Guide*, Academic Press, San Diego, CA, 1990