

# Extraction of Level Sets from 2D Images

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# 1 Introduction

For the purposes of this algorithm, a *2D image* is assumed to be a  $B_x \times B_y$  array of integer values. The pixel locations are  $(x, y)$  where  $0 \leq x < B_x$  and  $0 \leq y < B_y$ . No restriction is assumed on the pixel values  $I_{xy}$  other than they are integer-valued.

A continuous formulation of the image is required. For *domain square* with corners  $(x_0, y_0)$ ,  $(x_0 + 1, y_0)$ ,  $(x_0, y_0 + 1)$ , and  $(x_0 + 1, y_0 + 1)$ , let  $F(x, y)$  be a continuous function that represents the image on the square. It is not necessary that  $F$  match the pixel values at the four corners, but the two algorithms discussed in this document do have that property.

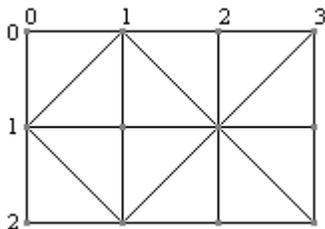
If  $L$  is a value in the range of  $F$ , then the *level set of  $F$  for  $L$*  is the set of points  $(x, y)$  that satisfy  $F(x, y) = L$ . Generally one expects the level sets to consist of curves, but isolated points are possible. For example, if  $F(x, y) = x^2 + y^2$ , the level set  $F(x, y) = L > 0$  consists of a single circle, but the level set  $F(x, y) = 0$  is a single point. The algorithms in this document construct both curves and points. A level set for the entire image is constructed from the level sets for the functions on the domain squares.

# 2 Extraction Using Linear Interpolation

The continuous formulation on domain squares is based on decomposing each square into two triangles and using linear interpolation on each triangle. Figure 2.1 shows a typical decomposition called a *symmetric triangulation*.

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**Figure 2.1** Symmetric triangulation of a  $4 \times 3$  image.



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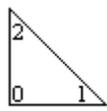
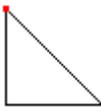
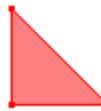
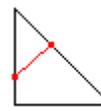
The triangulation of the upper left domain square is said to be *even*. Its two neighbors are said to be *odd*. Let the upper left corner of the domain square be  $(x_0, y_0)$ . The parity of the triangulation depends on the parity of the components. It is even whenever  $\text{Par}(x_0) = \text{Par}(y_0)$ , otherwise it is odd. The linear interpolant  $F(x, y)$  is defined for  $x_0 \leq x \leq x_0 + 1$  and  $y_0 \leq y \leq y_0 + 1$  as follows. Define  $F_{00} = F(x_0, y_0)$ ,  $F_{10} = F(x_0 + 1, y_0)$ ,

$F_{01} = F(x_0, y_0 + 1)$ , and  $F_{11} = F(x_0 + 1, y_0 + 1)$ . Define  $dx = x - x_0$  and  $dy = y - y_0$ ; then

$$F(x, y) = \left\{ \begin{array}{ll} F_{00} + (F_{10} - F_{00})dx + (F_{01} - F_{00})dy, & \text{Par}(x_0) = \text{Par}(y_0) \text{ and } dx + dy \leq 1 \\ F_{10} + F_{01} - F_{11} + (F_{11} - F_{01})dx + (F_{11} - F_{10})dy, & \text{Par}(x_0) = \text{Par}(y_0) \text{ and } dx + dy \geq 1 \\ F_{00} + (F_{10} - F_{00})dx + (F_{11} - F_{10})dy, & \text{Par}(x_0) \neq \text{Par}(y_0) \text{ and } dy \leq dx \\ F_{00} + (F_{11} - F_{01})dx + (F_{01} - F_{00})dy, & \text{Par}(x_0) \neq \text{Par}(y_0) \text{ and } dy \geq dx \end{array} \right\}.$$

Given a level value  $L$  and a triangle in the decomposition, each vertex value  $F$  satisfies  $F < L$ ,  $F = L$ , or  $F > L$ , for a total of 27 possibilities for the triangle. Each possibility corresponds to a triangle that has no level set points, a single level set point, a single level set line segment, or the function is zero over the entire triangle. The 27 possibilities can be partitioned into 6 topologically distinct cases, each case having at most 6 permutations. Table 2.1 shows the cases and permutations.

**Table 2.1** Cases and permutations for level sets of  $F$  on a triangle.

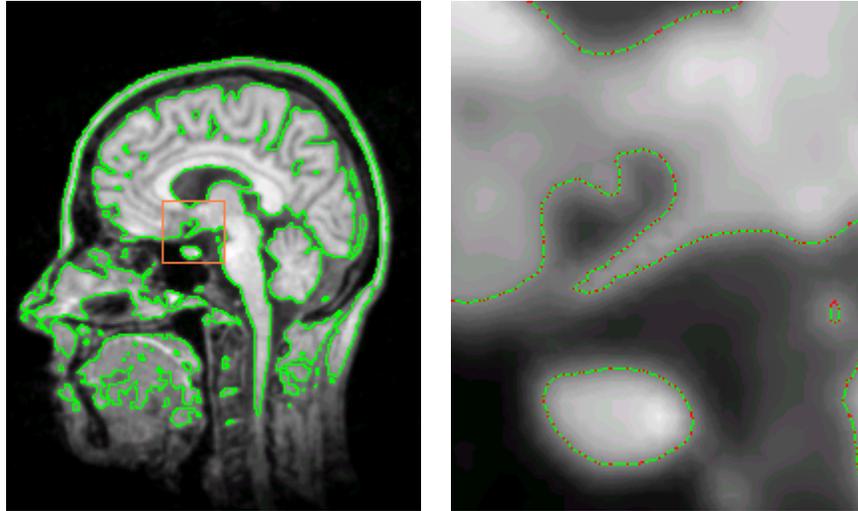
| case    | 0  | 1  | 2  | 3  | 4  | 5  |
|---------|--|--|--|--|--|--|
|         |  |  |  |  |  |  |
| permute |  |  |  |  |  |  |
| 0       | +++  | ++0  | +00  | 000  | ++-  | + - 0  |
| 1       | ---  | +0+  | 0+0  |  | + - +  | +0-  |
| 2       |  | 0++  | 00+  |  | - ++   | 0+ -   |
| 3       |  | --0  | -00  |  | --+  | -+0  |
| 4       |  | -0-  | 0-0  |  | -+-  | -0+  |
| 5       |  | 0--  | 00-  |  | +- -   | 0-+  |

The vertex ordering is shown in the figure for case 0 and is the same ordering for the figures of the remaining cases. Permutation 0 refers to that ordering. For example, case 5 and permutation 0 indicates that vertex 0 has positive value, vertex 1 has negative value, and vertex 2 has zero value. In case 0, the permutation is irrelevant since in either case there is no contribution to the level set. For cases 1, 2, 4, and 5, permutations 3 through 5 are effectively the same as permutations 0 through 2 for purposes of extracting level sets. The code makes this reduction. The total number of distinct possibilities is therefore 14.

Figure 2.2 shows a gray scale image with integers in the range  $[0, 1023]$ . The left image shows the original image superimposed with the vertex locations for level set 512. The right image shows a subimage superimposed with level sets. The edges from the level set extraction are drawn in green. The vertices are drawn in red.

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**Figure 2.2** Image and subimage with superimposed level sets from linear interpolation.



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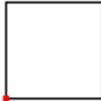
### 3 Extraction Using Bilinear Interpolation

The continuous formulation on domain squares is based on using each square as is and bilinearly interpolating. The linear interpolant  $F(x, y)$  is defined for  $x_0 \leq x \leq x_0 + 1$  and  $y_0 \leq y \leq y_0 + 1$  as follows. Define  $F_{00} = F(x_0, y_0)$ ,  $F_{10} = F(x_0 + 1, y_0)$ ,  $F_{01} = F(x_0, y_0 + 1)$ , and  $F_{11} = F(x_0 + 1, y_0 + 1)$ . Define  $dx = x - x_0$  and  $dy = y - y_0$ ; then

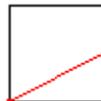
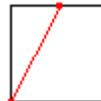
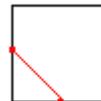
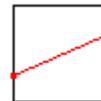
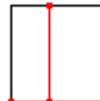
$$F(x, y) = F_{00}(1 - dx)(1 - dy) + F_{01}(1 - dx)dy + F_{10}dx(1 - dy) + F_{11}dxdy.$$

Given a level value  $L$  and a square in the decomposition, each vertex value  $F$  satisfies  $F < L$ ,  $F = L$ , or  $F > L$ , for a total of 81 possibilities for the square. These possibilities can be partitioned into topologically distinct cases, each case having at most 8 permutations. Tables 3.1, 3.2, and 3.3 show the cases and permutations.

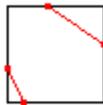
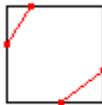
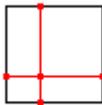
**Table 3.1** Some cases and permutations for level sets of  $F$  on a square.

| case    | 0   | 1   | 2   | 3   | 4   | 5   |
|---------|---|---|---|---|---|---|
| permute |  |  |  |  |  |  |
| 0       | ++++  | 0+++  | 00++  | 0+0+  | 000+  | 0000  |
| 1       | ----  | +0++  | +00+  | +0+0  | 00+0  |   |
| 2       |   | ++0+  | ++00  | 0-0-  | 0+00  |   |
| 3       |   | +++0  | 0++0  | -0-0  | +000  |   |
| 4       |   | 0---  | 00--  | 0+0-  | 000-  |   |
| 5       |   | -0--  | -00-  | +0-0  | 00-0  |   |
| 6       |   | --0-  | --00  | 0-0+  | 0-00  |   |
| 7       |   | -- -0   | 0--0  | -0+0  | -000  |   |

**Table 3.2** Some cases and permutations for level sets of  $F$  on a square.

| case    | 6   | 7   | 8   | 9   | 10  | 11  |
|---------|---|---|---|---|---|---|
| permute |  |  |  |  |  |  |
| 0       | 0+--  | 0++-  | +----   | ++--  | 0+-+  | 00+-  |
| 1       | -0+-  | -0++  | -+--  | -++-  | +0+-  | -00+  |
| 2       | --0+  | + -0+   | --+-  | --++  | -+0+  | + -00   |
| 3       | + - -0  | ++-0  | ----+   | +---+   | + - +0  | 0 + -0  |
| 4       | 0 - ++  | 0 - -+  | - + ++  |   | 0 - +-  | 00 - +  |
| 5       | +0 - +  | +0 - -  | + - ++  |   | -0 - +  | +00 -   |
| 6       | ++0-  | -+0-  | ++-+  |   | + - 0-  | -+00  |
| 7       | -+ +0   | -- +0   | +++-  |   | -+ -0   | 0 - +0  |

**Table 3.3** Last case and permutations for level sets of  $F$  on a square. The three cases depend on the relative values of the  $x$ -coordinates of the points.

| case    | 12  |   |  |
|---------|---|---|--|
| permute |  |  |  |
| 0       | + - + -   |   |  |
| 1       | - + - +   |   |  |

The vertex ordering is shown in the figure for case 0 and is the same ordering for the figures of the remaining cases. Permutation 0 refers to that ordering. For example, case 10 and permutation 0 indicates that vertex 0 has zero value, vertex 1 has positive value, vertex 2 has negative value, and vertex 3 has positive value. As in the linear interpolation, some permutations are effectively the same as others. The total number of distinct cases is 41.

Figure 3.1 shows a gray scale image with integers in the range  $[0, 1023]$ . The left image shows the original image superimposed with the vertex locations for level set 512. The right image shows a subimage superimposed with level sets. The edges from the level set extraction are drawn in green. The vertices are drawn in red.

**Figure 3.1** Image and subimage with superimposed level sets from bilinear interpolation.

