

Computing a Point of Reflection on a Sphere

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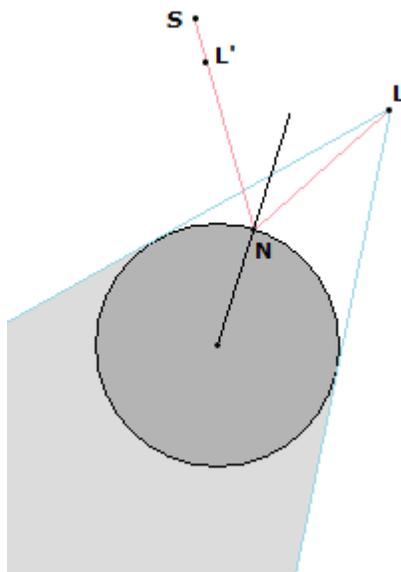
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1 Introduction

A point light is located at position \mathbf{L} , which is more than one unit of distance from the origin. A sphere of radius 1 and centered at the origin will reflect light rays from the point light. A point \mathbf{S} outside the sphere potentially receives a reflected ray of light. If it does, we wish to compute the point \mathbf{N} on the sphere at which the light ray is reflected to reach the point \mathbf{S} .

Imagine the smallest-angled single-sided cone whose vertex is \mathbf{L} and that contains the sphere. Figure 1.1 illustrates this.

Figure 1.1 The smallest-angled single-sided cone for the light position \mathbf{L} and the sphere. The point \mathbf{S} receives light as long as it is outside the sphere and not inside the shadowed area hidden from the light position. The point \mathbf{L}' is the reflection of \mathbf{L} through the ray whose origin is the sphere center and whose direction is \mathbf{N} .



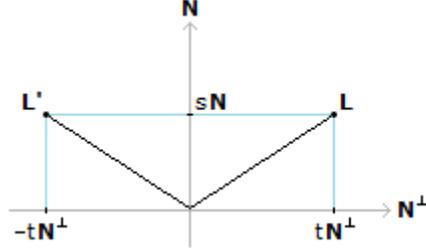
The point \mathbf{S} will receive a reflected light ray as long as it is outside the sphere and not in the shadow of the sphere, as illustrated in Figure 1.1. The point \mathbf{L}' is the reflection of \mathbf{L} through the ray whose origin is the sphere center and whose direction is \mathbf{N} .

The key observation is that \mathbf{N} must be chosen so that the vectors $\mathbf{S} - \mathbf{N}$ and $\mathbf{L}' - \mathbf{N}$ are parallel and in the same direction.

2 Reflection of a Vector

Figure 1.1 illustrates the reflection \mathbf{L}' of the point \mathbf{L} through the normal ray. We need to compute this reflection. Figure 2.1 helps to understand how to do this.

Figure 2.1 The reflection \mathbf{L}' of \mathbf{L} through a ray with direction \mathbf{N} .



The point \mathbf{L} has a component in the \mathbf{N} direction. After projecting out that component, the remainder is in some direction \mathbf{N}^\perp that is perpendicular to \mathbf{N} . Thus, we may write

$$\mathbf{L} = s\mathbf{N} + t\mathbf{N}^\perp \quad (1)$$

where $s = \mathbf{N} \cdot \mathbf{L}$. The reflection through \mathbf{N} amounts to changing sign on the perpendicular component,

$$\mathbf{L}' = s\mathbf{N} - t\mathbf{N}^\perp \quad (2)$$

The sum of the vectors in Equations (1) and (2) is

$$\mathbf{L} + \mathbf{L}' = 2s\mathbf{N} = 2(\mathbf{N} \cdot \mathbf{L})\mathbf{N} \quad (3)$$

We may solve for the reflected vector,

$$\mathbf{L}' = 2(\mathbf{N} \cdot \mathbf{L})\mathbf{N} - \mathbf{L} \quad (4)$$

3 Computing a Reflection Point

The assumption in this section is that \mathbf{S} and \mathbf{L} are not parallel vectors. For if they were, the point of reflection is $\mathbf{N} = \mathbf{L}/|\mathbf{L}|$; that is, the light is reflected in the opposite direction to reach \mathbf{S} . Our assumption has the consequence that $\mathbf{S} \times \mathbf{L} \neq \mathbf{0}$ (parallel vectors have a nonzero cross product).

Figure 1.1 shows that \mathbf{N} must bisect the rays whose common origin is \mathbf{N} and whose directions are $\mathbf{S} - \mathbf{N}$ and $\mathbf{L} - \mathbf{N}$. Therefore, we may represent

$$\mathbf{N} = x\mathbf{S} + y\mathbf{L} \quad (5)$$

for some scalars $x > 0$ and $y > 0$. Observe that

$$\mathbf{S} \times \mathbf{N} = y\mathbf{S} \times \mathbf{L}, \quad \mathbf{N} \times \mathbf{L} = x\mathbf{S} \times \mathbf{L}, \quad \mathbf{N} \cdot \mathbf{L} = x\mathbf{S} \cdot \mathbf{L} + y\mathbf{L} \cdot \mathbf{L} \quad (6)$$

Because \mathbf{N} is a unit-length vector, we also know that

$$1 = \mathbf{N} \cdot \mathbf{N} = x^2\mathbf{S} \cdot \mathbf{S} + 2xy\mathbf{S} \cdot \mathbf{L} + y^2\mathbf{L} \cdot \mathbf{L} \quad (7)$$

As noted in the introduction, \mathbf{N} must be chosen so that the vectors $\mathbf{S} - \mathbf{N}$ and $\mathbf{L}' - \mathbf{N}$ are parallel. Their cross product must be the zero vector,

$$\begin{aligned}
\mathbf{0} &= (\mathbf{S} - \mathbf{N}) \times (\mathbf{L}' - \mathbf{N}) \\
&= (\mathbf{S} - \mathbf{N}) \times [(2\mathbf{N} \cdot \mathbf{L} - 1)\mathbf{N} - \mathbf{L}] \\
&= (2\mathbf{N} \cdot \mathbf{L} - 1)\mathbf{S} \times \mathbf{N} - \mathbf{S} \times \mathbf{L} + \mathbf{N} \times \mathbf{L} \\
&= [(2\mathbf{N} \cdot \mathbf{L} - 1)y - 1 + x]\mathbf{S} \times \mathbf{L} \\
&= [(2x\mathbf{S} \cdot \mathbf{L} + 2y\mathbf{L} \cdot \mathbf{L} - 1)y - 1 + x]\mathbf{S} \times \mathbf{L}
\end{aligned} \tag{8}$$

where we have used Equations (2), (5), and (6). By assumption, $\mathbf{S} \times \mathbf{L} \neq \mathbf{0}$, so Equation (8) implies

$$(2x\mathbf{S} \cdot \mathbf{L} + 2y\mathbf{L} \cdot \mathbf{L} - 1)y - 1 + x = 0 \tag{9}$$

Equations (7) and (9) are two quadratic equations in two unknowns x and y ,

$$p(x, y) = ax^2 + 2bxy + cy^2 - 1 = 0, \quad q(x, y) = 2bxy + 2cy^2 + x - y - 1 = 0 \tag{10}$$

where $a = \mathbf{S} \cdot \mathbf{S}$, $b = \mathbf{S} \cdot \mathbf{L}$, and $c = \mathbf{L} \cdot \mathbf{L}$. We may solve $q(x, y) = 0$ for x in terms of y ,

$$x = \frac{-2cy^2 + y + 1}{2by + 1} \tag{11}$$

Substituting into the equation $p(x, y) = 0$, we have

$$\frac{a(-2cy^2 + y + 1)^2 + 2by(2by + 1)(-2cy^2 + y + 1) + (cy^2 - 1)(2by + 1)^2}{(2by + 1)^2} = 0 \tag{12}$$

The numerator of Equation (12) is the quartic polynomial,

$$r(y) = 4c(ac - b^2)y^4 - 4(ac - b^2)y^3 + (a + 2b + c - 4ac)y^2 + 2(a - b)y + a - 1 = 0 \tag{13}$$

Observe that

$$ac - b^2 = (\mathbf{S} \cdot \mathbf{S})(\mathbf{L} \cdot \mathbf{L}) - (\mathbf{S} \cdot \mathbf{L})^2 = |\mathbf{S} \times \mathbf{L}|^2 \neq 0 \tag{14}$$

so the coefficient of y^4 is not zero, which means $r(y)$ really has degree 4.

Now compute the real-valued roots of $r(y) = 0$. For each root $\bar{y} > 0$, compute $\bar{x} = (-2c\bar{y}^2 + \bar{y} + 1)/(2b\bar{y} + 1)$ from Equation (11). Of all the pairs (\bar{x}, \bar{y}) , select that pair for which $\bar{x} > 0$ and $\bar{y} > 0$. The point of reflection is $\mathbf{N} = \bar{x}\mathbf{S} + \bar{y}\mathbf{L}$.