## Game Mathematics

(12 Week Lesson Plan)

## Lesson 1: Set Theory

Textbook: Chapter One (pgs. 1-15)

## Goals:

We begin the course by introducing the student to a new vocabulary and set of rules that will be foundational to the mathematical discussions that follow in subsequent lessons. In this first lesson, students are introduced to Set Theory. The idea is to learn the basics of this allimportant branch of mathematics so that students are prepared to tackle and understand the concept of mathematical functions, which will play a major role in early lessons. Students will learn about how entities are grouped into sets and how to conduct various operations on those sets such as unions and intersections (i.e. the algebra of sets). We conclude with a brief introduction to the relationship between functions and sets to set the stage for the next lesson.

Key Topics:

Introduction to Set Theory

- The Language of Set Theory
- Set Membership
- Subsets, Supersets, and Equality
- The Algebra of Set Theory
- Set Theory and Functions

Projects: Varied Exercises

Exams/Quizzes: 5 Question Quiz (multiple choice)
Covers select topics from Chapter/ Lecture One $2 \%$ of final grade

Recommended Study Time (hours): 5-7

## Lesson 2: Functions

Textbook: $\quad$ Chapter Two (pgs. 19-47)

## Goals:

In this lesson, students are introduced to mathematical functions. We will begin by talking about the role of functions and look at the concept of mapping values between domain and range. From there we will spend a good deal of time looking at how to visualize various kinds of functions using graphs. This will set the stage for discussion of some of the most popular functions that are used in game development. We will begin with the absolute value function and then move on to discuss both exponential and logarithmic functions. Students will get an opportunity to see how these functions can be used to model various kinds of phenomena. From the fog used in games to calculating how players take weapon damage, students will see hands-on how such functions play a vital role in creating interesting effects. One of the more important things we will do is go step by step through the process of designing a function and coming up with a means for selecting appropriate values that reflect the desired outcomes.

## Key Topics:

Mathematical Functions

- Graphs
o Single-Variable Functions
o Two-Variable Functions
- Families of Functions
o Absolute Value Function
o Exponential Functions
- Fog Density
- Damage Calculations
o Logarithmic Functions
- Using the Log Function for Game Development

Projects: Varied Exercises

Exams/Quizzes: 5 Question Quiz (multiple choice)
Covers select topics from Chapter/ Lecture Two $2 \%$ of final grade

Recommended Study Time (hours): 5-7

## Lesson 3: Polynomials

Textbook: $\quad$ Chapter Three (pgs. 49-71)

## Goals:

In this lesson, students will learn about polynomials. We will begin with an examination of the algebra of polynomials, and then move on to look at the graphs for various kinds of polynomial functions. Once our theoretical discussions are concluded, we will focus on the application of different kinds of polynomials in game development projects. We start with linear interpolation using polynomials that is commonly used to draw polygons on the display. From there we will look at how to take complex functions that would be too costly to compute in a real-time game environment and use polynomials to approximate the behavior of the function to produce similar results. We will wrap things up by looking at how polynomials can be used as a means for predicting the future values of variables, which can be useful under a number of different game scenarios (such as managing network packet latency for example).

## Key Topics:

Polynomials

- Polynomial Algebra (Single Variable)
o Addition/Subtraction
o Scalar Multiplication
o Multiplication/Division
- Quadratic Equations
- Graphing Polynomials
- Using Polynomials
o Linear Interpolation
o Approximating Functions
o Prediction

Projects: Varied Exercises

Exams/ Quizzes: 5 Question Quiz (multiple choice)
Covers select topics from Chapter/ Lecture Three $2 \%$ of final grade

Recommended Study Time (hours): 5-7

## Lesson 4: Basic Trigonometry I

Textbook: Chapter Four (pgs. 75-97)

## Goals:

Triangles are the core primitive of most modern 3D game engines. As such it is vital that students have a firm grasp of the properties of triangles, and right triangles in particular. In this lesson, students will get a crash course in some of the core elements of trigonometry. We will talk about the properties of triangles and look at the relationships that exist between their internal angles and the lengths of their sides. This will lead to discussion of the most commonly used trigonometric functions that relate triangle properties to unit circles. This includes the sine, cosine and tangent functions. We will use these properties and functions to solve a number of issues related to graphics programming, such as modeling an animated wave function such as might be used for water or cloth simulation, and also look at how to use these concepts to render circles and ellipses on the display.

## Key Topics:

Angles

- Common Angles
- The Polar Coordinate System

Triangles

- Properties
- Right Triangles

Introduction to Trigonometry

- The Trigonometric Functions
- Applications of Basic Trigonometry
- Solving Triangle Problems

Modeling Phenomena

- Modeling Waves
- Drawing Circles and Ellipses
- Projection


## Projects: Varied Exercises

Exams/ Quizzes: 5 Question Quiz (multiple choice)
Covers select topics from Chapter/ Lecture Four $2 \%$ of final grade

Recommended Study Time (hours): 6-8

## Lesson 5: Basic Trigonometry II

Textbook: Chapter Five (pgs. 101-122)

## Goals:

Picking up where the last lesson left off, students continue their examination of trigonometric functions. In this lesson, we will look at the very important inverse trig functions such as arcsin, arcos, and arctan, and see how they can be used to determine angle values. We will also introduce students to the core trig identities such as the reduction and double angle identities and use them as a means for deriving proofs. Being able to derive proofs are an important part of the mathematicians skill set and we will begin to become more formal with this concept in this lesson. As usual, we will look at applications to game technology and see how trig functions can be used to rotate points in two and three dimensions and also how to construct a proper field of view (FOV) for an in-game camera system.

## Key Topics:

Trig Functions

- Derivative Trigonometric Functions
- Inverse Trig Functions
- Identities
o Pythagorean Identities
o Reduction Identities
o Angle Sum/ Difference Identities
o Double-Angle Identities
o Sum-To-Product Identities
o Product-to-Sum Identities
o Triangle Laws
Applications
- Point Rotation
- Field-of-View

Projects: Varied Exercises
Exams/Quizzes: 5 Question Quiz (multiple choice)
Covers select topics from Chapter/ Lecture Five $2 \%$ of final grade

Recommended Study Time (hours): 6-8

## Lesson 6: Analytic Geometry I

Textbook: Chapter Six (pgs. 125-149)

## Goals:

Beyond triangles, students will also need to understand other important constructs. In this lesson, we will introduce analytic geometry as the means for using functions and polynomials to mathematically represent points, lines, planes and ellipses. All of these concepts are vital in game development since they are used in rendering and optimization, collision detection and response, game physics, and other critical areas. We will start with points in space and move on to simple 2D lines and their various forms (including the all-important parametric representation). We will look at intersection formulas and distance formulas with respect to lines, points, and planes and also briefly talk about ellipsoidal intersections.

## Key Topics:

Points and Lines

- Two-Dimensional Lines
- Parametric Representation
- Parallel and Perpendicular Lines
- Intersection of Two Lines
- Distance from a Point to a Line
- Angles between Lines
- Three-Dimensional Lines

Ellipses and Ellipsoids

- Intersecting Lines with Ellipses
- Intersecting Lines with Spheres

Planes

- Intersecting Lines with Planes

Projects: Varied Exercises

Exams/Quizzes: 5 Question Quiz (multiple choice) Covers select topics from Chapter/ Lecture Six $2 \%$ of final grade

Recommended Study Time (hours): 5-7

## Lesson 7: Vector Mathematics

Textbook: Chapter Seven (pgs. 151-174)

## Goals:

In this lesson, students are introduced to vector mathematics - the core of the 3D graphics engine. After an introduction to the concept of vectors, we will look at how to perform various important mathematical operations on them. This will include addition and subtraction, scalar multiplication, and the all-important dot and cross products. After laying this computational foundation, we will look at the use of vectors in games and talk about their relationship with planes and the plane representation, revisit distance calculations using vectors and see how to rotate and scale geometry using vector representations of mesh vertices.

## Key Topics:

Elementary Vector Math

- Linear Combinations
- Vector Representations
- Addition/Subtraction
- Scalar Multiplication/ Division
- Vector Magnitude
- The Dot Product
- Vector Projection
- The Cross Product

Applications of Vectors

- Directed Lines
- Vectors and Planes
o Back-face culling
o Vector-based Plane Representation
- Distance Calculations (Points, Planes, Lines)
- Point Rotation, Scaling, Skewing

Projects: Varied Exercises

Exams/Quizzes: 5 Question Quiz (multiple choice)
Covers select topics from Chapter/ Lecture Seven $2 \%$ of final grade

Recommended Study Time (hours): 8-10

## Lesson 8: Matrix Mathematics I

Textbook: $\quad$ Chapter Eight (pgs. 177-188)

## Goals:

In this lesson, students are introduced to the concept of a matrix. Like vectors, matrices are one of the core components of every 3D game engine and as such are required learning. In this first of two lessons, we will look at matrices from a purely mathematical perspective. We will talk about what matrices are and what problems they are intended to solve and then we will look at various operations that can be performed using them. This will include topics like matrix addition and subtraction and multiplication by scalars or by other matrices. We will conclude the lesson with an overview of the concept of using matrices to solve systems of linear equations. We will do this by lightly touching on the notion of Gaussian elimination.

## Key Topics:

## Matrices

- Matrix Relations
- Matrix Operations
o Addition/Subtraction
o Scalar Multiplication
o Matrix Multiplication
o Transpose
o Determinant
o Inverse
- Systems of Linear Equations
o Gaussian Elimination

Projects: Varied Exercises

Exams/ Quizzes: 5 Question Quiz (multiple choice)
Covers select topics from Chapter/ Lecture Eight $2 \%$ of final grade

Recommended Study Time (hours): 8-10

Textbook: $\quad$ Chapter Nine (pgs. 191-210)

## Goals:

In this lesson, we continue our discussion of matrix mathematics and introduce the student to the problem that matrices are generally used to solve in 3D games: transformations. After introducing the idea of linear transformations, we will take a brief detour to examine how an important non-linear operation like translation (used to reposition points in 3D game worlds) can be made compliant with our matrix operations by introducing 4D homogenous coordinates. Once done, we will examine a number of common matrices used to effect transformations in 3D games. This will include projection, translation, scaling and skewing, as well as rotations around all three coordinate axes. We will wrap up with the actual vector/matrix transformation operation (multiplication) which represents the foundation of the 3D graphics rendering pipeline.

## Key Topics:

Linear Transformations

- Computing Linear Transformation Matrices
- Translation and Homogeneous Coordinates
- Transformation Matrices o The Scaling Matrix
o The Skewing Matrix
o The Translation Matrix
o The Rotation Matrices
o The Projection Matrix
- Linear Transformations in 3D Games

Projects: Varied Exercises

Exams/ Quizzes: 5 Question Quiz (multiple choice)
Covers select topics from Chapter/ Lecture Nine $2 \%$ of final grade

Recommended Study Time (hours): 10-12

Textbook: Chapter Ten (pgs. 211-227)

## Goals:

In this lesson, students are introduced to quaternion mathematics. To set the stage for quaternions, which are hyper-complex numbers, we will first examine the concept of imaginary numbers and look at the various arithmetical operations that can be performed on them. We will look at the similarities and differences with respect to the real numbers. Once done, we will introduce complex numbers and again look at the algebra involved. Finally we will examine the quaternion and its associated algebra. With the formalities out of the way we will look at applications of the quaternion in game development. Primarily the focus will be on how to accomplish rotations about arbitrary axes and how to solve the gimbal lock problem encountered with Euler angles. We put this concept to use to create an updated world to view space transformation matrix that is derived from a quaternion after rotation has taken place.

Key Topics:
Imaginary Numbers

- Powers
- Multiplication/Division
- Addition/ Subtraction

Complex Numbers

- Addition/ Subtraction
- Multiplication/Division
- Powers
- Complex Conjugates
- Magnitude

Quaternions

- Addition/ Subtraction
- Multiplication
- Complex Conjugates
- Magnitude
- Inverse
- Rotations
- World-to-View Transformation

Projects: Varied Exercises
Exams/Quizzes: 5 Question Quiz (multiple choice)
Covers select topics from Chapter/ Lecture Ten $2 \%$ of final grade
Recommended Study Time (hours): 10-12

## Lesson 11: Analytic Geometry II

Textbook: Chapter Eleven (pgs. 229 - 251)

## Goals:

In this lesson, we will focus on some of the practical applications of mathematics. In this particular case we will look at how analytic geometry plays an important role in a number of different areas of game development. We will start by looking at how to design a simple collision/response system in 2D using lines and planes as a means for modeling a simple billiards simulation. We will continue our intersection discussion by looking at a way to detect collision between two convex polygons of arbitrary shape. From there we will see how to use vectors and planes to create reflections such as might be seen in a mirror. Then we will talk about the use of a convex volume to create shadows in the game world. Finally we will wrap things up with a look at the Lambertian diffuse lighting model to see how vector dot products can be used to determine the lighting and shading of points across a surface.

## Key Topics:

- 2D Collisions
- Reflections
- Polygon/Polygon Intersection
- Shadow Casting
- Lighting

Projects: Varied Exercises
Exams/Quizzes: NONE
Recommended Study Time (hours): 8-10

Lesson 12: Exam Preparation and Course Review
Textbook: NONE

## Goals:

In this final lesson we will leave the student free to prepare for and take their final examination. Multiple office hours will be held for student questions and answers.

Key Topics: NONE

Projects: NONE

Exams/Quizzes: NONE

Recommended Study Time (hours): 15-20

## Final Examination

The final examination in this course will consist of 25 multiple-choice and true/false questions pulled from the first 10 textbook chapters. Students are encouraged to use the lecture presentation slides as a means for reviewing the key material prior to the examination. The exam should take no more than three hours to complete. It is worth $80 \%$ of student final grade.

